NYCMT 2024-2025 Homework #4 Solutions

NYCMT

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Problem 1. Prove that if n is a nonnegative integer, then $19 \cdot 8^n + 17$ is not a prime number.

Solution. First, we take the equation mod 3. Notice that

$$19 \cdot 8^n + 17 \equiv (-1)^n - 1 \pmod 3,$$

so $19 \cdot 8^n + 17$ is divisible by 3 when n is even. Next, we take the equation mod 5. Since

$$19 \cdot 8^n + 17 \equiv -3^n + 2 \pmod 5,$$

and $3^4 \equiv 1 \pmod 5$, we can see that the expression is periodic mod 4. Noting that $-3^3 + 2 = -25 \equiv 0 \pmod 5$, we have that $19 \cdot 8^n + 17 \equiv 0 \pmod 5$ when $n \equiv 3 \pmod 4$. Finally, we take the equation mod 13. Since

$$19 \cdot 8^n + 17 \equiv 6 \cdot 8^n + 4 \pmod{13}$$

and $8^4 \equiv 1 \pmod{13}$, we can see that the expression is periodic mod 4. Noting that $6 \cdot 8^1 + 4 = 52 \equiv 0 \pmod{13}$, we have that $19 \cdot 8^n + 17 \equiv 0 \pmod{13}$ when $n \equiv 1 \pmod 4$.

Since $19 \cdot 8^n + 17$ is always divisible by 3, 5, or 13 (and is never equal to any of these, which we can show by noting that it is increasing), it can never be prime when n is a nonnegative integer. □

Problem 2. Find all positive integers $n \geq 2$ for which one can fill in the cells of an $n \times n$ grid with the numbers 0, 1, 2 such that, when calculating the sum of the numbers in each row and each column, the numbers $1, 2 \ldots, 2n$ are obtained in some order.

Answer. Any even n

Solution. First, we claim that n cannot be odd. Let r_i denote the sum of all the numbers in the cells of the ith row, and c_i denote the sum of all the numbers in the cells of the ith column. Then

$$\sum_{i=1}^{n} r_i = \sum_{i=1}^{n} c_i,$$

since they are both equal to the sum of all the numbers in the grid. Thus,

$$\sum_{i=1}^{n} r_i + \sum_{i=1}^{n} c_i = 1 + 2 + \cdots + 2n = n(2n+1)$$

must be even, which is impossible for odd n.

Next, we will show that any even n works. Let a_{ij} denote the entry in the ith row from the bottom and jth column from the left. We claim that the following construction satisfies the desired property:

- If $i > j$, $a_{ij} = 0$.
- If $i < j$, $a_{ij} = 2$.
- If $i = j \leq \frac{n}{2}$, $a_{ij} = 1$.
- If $i = j > \frac{n}{2}$, $a_{ij} = 2$.

For example, the following is the construction for $n = 8$.

0	0	0	0	0	0	0	2
0	0	0	0	0	0	2	2
0	0	0	0	0	2	2	2
0	0	0	0	2	2	2	2
0	0	0	1	2	2	2	2
0	0	1	2	2	2	2	2
0	1	2	2	2	2	2	2
1	2	2	2	2	2	2	2

To show this works, net $n = 2k$, and want to show that in the sums of all the rows and columns, the numbers $1, 2, \ldots, 4k$ appear in some order. We inspect four cases:

1. Rows r_1 through r_k:
   Let i be the index of the row. Then there must be $2k - i$ cells in the row filled with 2, 1 cell filled with 1, and $i - 1$ cells filled with 0. Thus, these row sums can be represented by the set
   $$\{2(2k - i) + 1 \mid 1 \leq i \leq k\},$$
   or $\{2k + 1, 2k + 3, \ldots, 4k - 1\}$.

2. <u>Rows r_{k+1} through r_{2k}:</u>
   Let i be the index of the row. Then there must be $2k - i + 1$ cells in the row filled with 2 and $i - 1$ cells filled with 0. Thus, these row sums can be represented by the set

   $$\{2(2k - i + 1) \mid k + 1 \leq i \leq 2k\},$$

   or $\{2, 4, \ldots, 2k\}$.

3. <u>Columns c_1 through c_k:</u>
   Let i be the index of the row. Then there must be $2k - i$ cells in the row filled with 0, 1 cell filled with 1, and $i - 1$ cells filled with 2. Thus, these column sums can be represented by the set

   $$\{2i - 1 \mid 1 \leq i \leq k\},$$

   or $\{1, 3, \ldots, 2k - 1\}$.

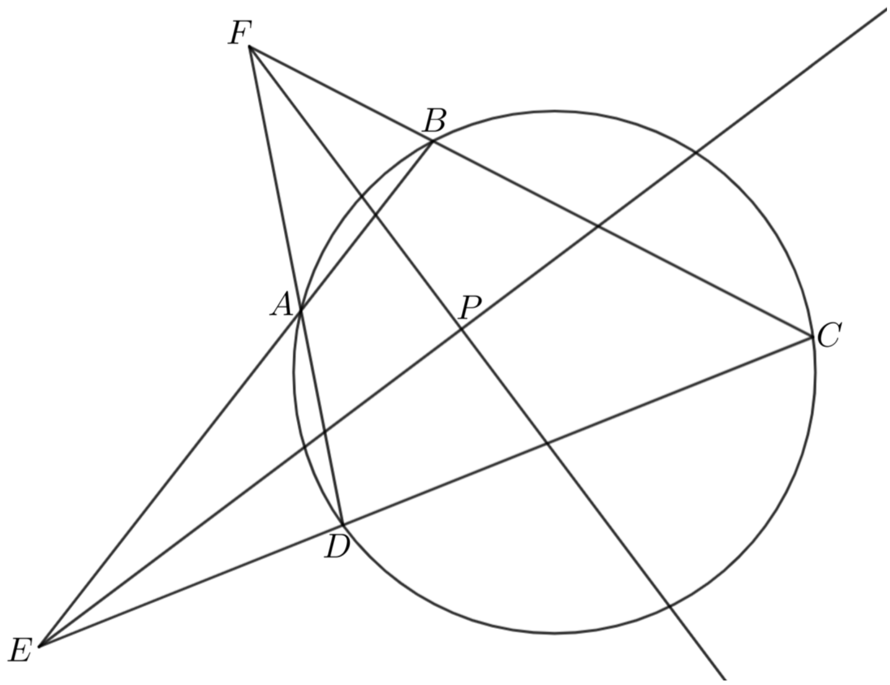
4. <u>Columns c_{k+1} through c_{2k}:</u>
   Let i be the index of the row. Then there must be $2k - i$ cells in the row filled with 0 and i cells filled with 2. Thus, these column sums can be represented by the set

   $$\{2i \mid k + 1 \leq i \leq 2k\},$$

   or $\{2k + 2, 2k + 4, \ldots, 4k\}$.

Taking the union of all these sets gives us all the numbers from 1 to $4k$, inclusive, as desired. □

Problem 3. Let $ABCD$ be a cyclic quadrilateral such that no two sides are parallel to each other. Let lines AB and CD intersect at E, and let lines AD and BC intersect at F. Prove that the angle bisectors of $\angle AFB$ and $\angle BEC$ are perpendicular to each other.



Solution. Let P be the intersection of the angle bisectors of $\angle AFB$ and $\angle BEC$. We wish to show that $\angle FPE = 90°$.

We do this by inspecting the angle measures of $AFPE$. Notice that

$$\angle FAE = 180° + FAB = 180° + \angle BCD.$$

Next, we find the angle measures of $\angle AFP$ and $\angle PEA$. We see that

$$\angle PEA = \frac{1}{2}\angle CEB = \frac{1}{2}(180° - \angle BCD - \angle ABC)$$

and

$$\angle AFP = \frac{1}{2}\angle DFC = \frac{1}{2}(180° - \angle ADC - \angle BCD) = \frac{1}{2}(\angle ABC - \angle BCD).$$

Thus,

$$\begin{aligned}\angle FPE &= 360° - \angle FAE - \angle PEA - \angle AFP \\ &= 360° - (180° + \angle BCD) - \frac{1}{2}(180° - \angle BCD - \angle ABC) - \frac{1}{2}(\angle ABC - \angle BCD) \\ &= 90° - \angle BCD + \frac{1}{2}\angle BCD + \frac{1}{2}\angle ABC - \frac{1}{2}\angle ABC + \frac{1}{2}\angle BCD \\ &= 90°,\end{aligned}$$

so $\angle FPE = 90°$, as desired. $\square$

Problem 4. Does there exist a sequence of 2025 consecutive positive integers such that the kth term is divisible by $2026 - k$ for all $1 \leq k \leq 2025$?

Answer. $\boxed{\text{Yes}}$

Solution. Let n be the smallest of the 2025 consecutive positive integers. Then, we want the following congruence to hold for all integers $0 \leq k \leq 2024$:

$$\begin{aligned} n + k &\equiv 0 \pmod{(2025 - k)} \\ n &\equiv -k \pmod{(2025 - k)} \\ n &\equiv -2025 \pmod{(2025 - k)} \end{aligned}$$

where the third congruence is achieved by subtracting $(2025 - k)$ from the second congruence. Then, $n = -2025 + \text{lcm}(1, 2, 3, \ldots, 2025)$ is a solution.

Remark. More generally, the existence criteria for a solution to a system of modular congruences whose moduli are not pairwise coprime requires that, for each pair of congruences $n \equiv r_i \pmod{m_i}$ and $n \equiv r_j \pmod{m_j}$, the following holds:

$$r_i \equiv r_j \pmod{\gcd(m_i, m_j)}.$$

A proof of this result is left at the end of this remark. Applying this to the problem at hand, it suffices to show that $-i \equiv -j \pmod{\gcd(2025 - i, 2025 - j)}$. This is clear, since, by the Euclidean Algorithm,

$$\gcd(2025 - i, 2025 - j) \mid (2025 - i) - (2025 - j) = j - i$$

and $-i \equiv -j \pmod{(j - i)}$ is true.

Proof of existence criteria. It suffices to show the result for two congruences, as the general criteria arises from induction on the number of congruences. We would like to show that, for the following system of two congruences:

$$\begin{aligned} n &\equiv r_1 \pmod{m_1} \\ n &\equiv r_2 \pmod{m_2} \end{aligned}$$

a solution for $n \pmod{\text{lcm}(m_1, m_2)}$ exists $\Leftrightarrow r_1 \equiv r_2 \pmod{\gcd(m_1, m_2)}$.

Let $g = \gcd(m_1, m_2)$. We first prove the forward direction. If a solution for $n \pmod{\text{lcm}(m_1, m_2)}$ exists, let it be s. Then, $s \equiv r_1 \pmod{m_1}$ and $s \equiv r_2 \pmod{m_2}$. By the definition of gcd, $g \mid m_1, m_2$, so it is also true that $s \equiv r_1 \pmod{g}$ and $s \equiv r_2 \pmod{g}$. By transitivity, it is clear that $r_1 \equiv r_2 \pmod{g}$.

To prove the reverse direction, we first split each congruence into its own system of congruences based on the prime factorization of its modulus. That is, if $m_1 = \prod_{i=1}^{k} p_i^{e_i}$ for distinct primes p_i and integer exponents e_i, then we instead consider $n \equiv r_1 \pmod{p_i^{e_i}}$ for all $1 \leq i \leq k$. By the Chinese Remainder Theorem on pairwise coprime moduli, the original congruence and this resulting system are equivalent. We do the same with $m_2 = \prod_{j=1}^{l} q_j^{f_j}$ for distinct primes q_j and integer exponents f_j.

The reverse direction should now be clear; if $r_1 \equiv r_2 \pmod{\gcd(m_1, m_2)}$, then all congruences, specifically the ones where $p_i = q_j$ for some i and some j, are consistent. After discarding redundant congruences, the Chinese Remainder Theorem may be applied on the resulting system, as all moduli are pairwise coprime. $\square$

Problem 5. Let $P(x)$ be a polynomial with degree at most 8 such that for $k = 0, 1, \dots, 8$,

$$P(k) = \begin{cases} 0 & k \equiv 0 \pmod 3 \\ 1 & k \equiv 1 \pmod 3 \\ 2 & k \equiv 2 \pmod 3. \end{cases}$$

Find $P(9)$.

Answer. $\boxed{-81}$

Solution. Consider the polynomial $Q(x) = P(x) - x$. For $k = 0, 1, \dots, 8$,

$$Q(k) = \begin{cases} 0 & 0 \le k \le 2 \\ -3 & 3 \le k \le 5 \\ -6 & 6 \le k \le 8. \end{cases}$$

Since $\deg(P) \le 8$, $\deg(Q) \le 8$. Using finite differences, the 9th order finite difference

$$\Delta_1^9[Q](0) = \binom{9}{9}Q(9) - \binom{9}{8}Q(8) + \cdots + (-1)^9\binom{9}{0}Q(0) = 0$$

since $\deg(Q) \le 8$. Rearranging to solve for $Q(9)$, we get

$$Q(9) = \binom{9}{8}Q(8) - \binom{9}{7}Q(7) + \cdots + (-1)^8\binom{9}{0}Q(0).$$

Now substituting the values of $Q(k)$ for $k = 0, 1, \dots, 8$,

$$Q(9) = -6 \cdot \left(\binom{9}{8} - \binom{9}{7} + \binom{9}{6}\right) + 3 \cdot \left(\binom{9}{5} - \binom{9}{4} + \binom{9}{3}\right).$$

Since $\binom{9}{4} = \binom{9}{5}$, we can rewrite the above expression and solve:

$$Q(9) = -6 \cdot \left(\binom{9}{8} - \binom{9}{7} + \binom{9}{6}\right) + 3 \cdot \binom{9}{3}$$
$$Q(9) = -6 \cdot (9 - 36 + 84) + 3 \cdot 84 = -90.$$

Then, to find $P(9)$, $Q(x) = P(x) - x$ so $P(9) = Q(9) + 9 = -90 + 9 = \boxed{-81}$. □