OMO Fall 2015
Official Solutions

1. Evalute
$$\sqrt{\binom{8}{2}+\binom{9}{2}+\binom{15}{2}+\binom{16}{2}}.$$

Proposed by Evan Chen.

Answer. $\boxed{17}$.

Solution. The main observation is that
$$\binom{8}{2}+\binom{9}{2}=8^2 \qquad \binom{15}{2}+\binom{16}{2}=15^2$$
so that the desired sum is $\sqrt{8^2+15^2}=17$, a well-known Pythagorean triple. □

2. At a national math contest, students are being housed in single rooms and double rooms; it is known that 75% of the students are housed in double rooms. What percentage of the rooms occupied are double rooms?

Proposed by Evan Chen.

Answer. $\boxed{60}$.

Solution. Assume there are k students in single rooms and $3k$ students in double rooms. The number of single rooms is k, and the number of double rooms is $\frac{3}{2}k$. So the answer is $\frac{\frac{3}{2}k}{\frac{3}{2}k+k}=\frac{3}{5}$, which is 60%. □

3. How many integers between 123 and 321 inclusive have exactly two digits that are 2?

Proposed by Yannick Yao.

Answer. $\boxed{18}$.

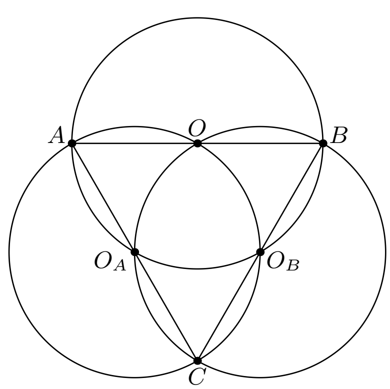
Solution. These integers must take on the form $\overline{22n}$ or $\overline{2n2}$, where $n \neq 2$. Since there are 9 choices for n in each case, the answer is $2 \cdot 9 = 18$. □

4. Let ω be a circle with diameter AB and center O. We draw a circle ω_A through O and A, and another circle ω_B through O and B; the circles ω_A and ω_B intersect at a point C distinct from O. Assume that all three circles ω, ω_A, ω_B are congruent. If $CO=\sqrt{3}$, what is the perimeter of $\triangle ABC$?

Proposed by Evan Chen.

Answer. $\boxed{6}$.

Solution. Let O_A be the center of ω_A and O_B the center of ω_B. Notice that O_A and O_B must lie on the same sides of line AB, since the assumptions of the problem implicitly tell us the circles are not tangent at O.



Moreover, by symmetry we have that $\overline{OC} \perp \overline{AB}$; so O_A and O_B are the midpoints of AC and BC. In particular, AOO_A and BOO_B are equilateral; finally we deduce ABC is equilateral too.

Since $OC = \sqrt{3}$, we find $AB = BC = CA = 2$, so the perimeter is 6. □

5. Merlin wants to buy a magical box, which happens to be an n-dimensional hypercube with side length 1 cm. The box needs to be large enough to fit his wand, which is 25.6 cm long. What is the minimal possible value of n?

   Proposed by Evan Chen.

   Answer. $\boxed{656}$.

   Solution. By the Pythagorean Theorem in n-dimensional space, the maximal length is given by the diagonal

$$\sqrt{\underbrace{(1-0)^2 + (1-0)^2 + \cdots + (1-0)^2}_{n \text{ times}}} = \sqrt{n}.$$

   This is the distance from $(0, \ldots, 0)$ to $(1, \ldots, 1)$ as points in $\mathbb{R}^n$. So we require $\sqrt{n} \ge 25.6 \iff n \ge 655.36$; the smallest integer n is $n = 656$. □

6. Farmer John has a (flexible) fence of length L and two straight walls that intersect at a corner perpendicular to each other. He knows that if he doesn't use any walls, he can enclose a maximum possible area of A_0, and when he uses one of the walls or both walls, he gets a maximum of area of A_1 and A_2 respectively. If $n = \frac{A_1}{A_0} + \frac{A_2}{A_1}$, find $\lfloor 1000n \rfloor$.

   Proposed by Yannick Yao.

   Answer. $\boxed{4000}$.

   Solution. It's clear that A_0 is achieved when the fence becomes a circle, A_1 being a half-circle, and A_2 a quarter circle. Then it becomes clear (after some easy computations) that $A_1 = 2A_0, A_2 = 2A_1$, and $n = 4$ so the answer becomes 4000. □

7. Define sequence $\{a_n\}$ as following: $a_0 = 0$, $a_1 = 1$, and $a_i = 2a_{i-1} - a_{i-2} + 2$ for all $i \ge 2$. Determine the value of a_{1000}.

   Proposed by Yannick Yao.

   Answer. $\boxed{1000000}$.

   Solution. We claim that in fact $a_n = n^2$ for every integer n. The proof is by induction on n with the base cases $n = 0$ and $n = 1$ given; for the inductive step, we observe that

$$2(i-1)^2 - (i-2)^2 + 2 = i^2$$

   as desired. Therefore $a_{1000} = 1000^2 = 1000000$. □

8. The two numbers 0 and 1 are initially written in a row on a chalkboard. Every minute thereafter, Denys writes the number $a + b$ between all pairs of consecutive numbers a, b on the board. How many odd numbers will be on the board after 10 such operations?

   Proposed by Michael Kural.

   Answer. $\boxed{683}$.

Solution. All numbers are integers at all points, so we will tacitly take modulo 2 everywhere. We claim that after k operations, the numbers on the board are

$$\underbrace{011011011\dots011}_{\frac{2^k-1}{3}\text{ blocks}}01$$

when k is even and

$$\underbrace{011011011\dots011}_{\frac{2^k+1}{3}\text{ blocks}}$$

when k is odd. (Note that in total, there are 2^k+1 numbers written.)

The proof of this observation is a direct induction on $k \ge 0$. Applying this to $k = 10$, we see the number of odd numbers is $\frac{2^{10}-1}{3} \cdot 2 + 1 = 683$. □

9. Let $s_1, s_2, \dots$ be an arithmetic progression of positive integers. Suppose that

$$s_{s_1} = x + 2, \quad s_{s_2} = x^2 + 18, \quad \text{and} \quad s_{s_3} = 2x^2 + 18.$$

Determine the value of x.

Proposed by Evan Chen.

Answer. $\boxed{16}$.

Solution. The main observation is that s_{s_1}, s_{s_2}, s_{s_3} must be in arithmetic progression since s_1, s_2, and s_3 are. From this, we have that $x+2$, x^2+18 and $2x^2+18$ are in arithmetic progression, hence $2(x^2+18) = (2x^2+18) + (x+2)$ which gives $x = 16$ immediately.

In fact, the sequence in question is $s_n = 16n - 14$. □

10. For any positive integer n, define a function f by

$$f(n) = 2n + 1 - 2^{\lfloor \log_2 n \rfloor + 1}.$$

Let f^m denote the function f applied m times.. Determine the number of integers n between 1 and 65535 inclusive such that $f^n(n) = f^{2015}(2015)$.

Proposed by Yannick Yao.

Answer. $\boxed{8008}$.

Solution. By observing the base-2 expansion of the integer, we see that the function is equivalent to removing the frontmost nonzero digit (which is 1) and adding a 1 at the end. Thus $f^n(n) = 2^{s(n)} - 1$, where $s(n)$ is the sum of binary digits of n. Since $2015 = 11111011111_2$ has $s(2015) = 10$, Therefore it suffices to find the number of positive integers with at most 16 binary digits exactly 10 of which are 1. This is $\binom{16}{10} = 8008$. □

11. A trapezoid $ABCD$ lies on the xy-plane. The slopes of lines BC and AD are both $\frac{1}{3}$, and the slope of line AB is $-\frac{2}{3}$. Given that $AB = CD$ and $BC < AD$, the absolute value of the slope of line CD can be expressed as $\frac{m}{n}$, where m, n are two relatively prime positive integers. Find $100m + n$.

Proposed by Yannick Yao.

Answer. $\boxed{1706}$.

Solution. The slope is $\frac{17}{6}$. Should be easy by a bit of coordinate bash or LOC vector form. □

12. Let a, b, c be the distinct roots of the polynomial $P(x) = x^3 - 10x^2 + x - 2015$. The cubic polynomial $Q(x)$ is monic and has distinct roots $bc - a^2$, $ca - b^2$, $ab - c^2$. What is the sum of the coefficients of Q?

    Proposed by Evan Chen.

    Answer. $\boxed{2015000}$.

    Solution. Considering the factorization of Q, we seek to compute $(1-bc+a^2)(1-ca+b^2)(1-ab+c^2)$. Since $1 = ab + bc + ca$ by Vieta's Formulas, this rewrites as

    $$(a(a+b+c))(b(a+b+c))(c(a+b+c)) = abc(a+b+c)^3 = 2015000. \qquad \square$$

13. You live in an economy where all coins are of value $1/k$ for some positive integer k (i.e. $1, 1/2, 1/3, \dots$). You just recently bought a coin exchanging machine, called the *Cape Town Machine*. For any integer $n > 1$, this machine can take in n of your coins of the same value, and return a coin of value equal to the sum of values of those coins (provided the coin returned is part of the economy). Given that the product of coins values that you have is 2015^{-1000}, what is the maximum number of times you can use the machine over all possible starting sets of coins?

    Proposed by Yang Liu.

    Answer. $\boxed{308}$.

    Solution. When you put n coins in the machine, the product multiplies by n^n times some more constants. It's obvoiusly optimal to choose n to be prime, so prime factorize 2015 and do some division. $\square$

14. Let $a_1, a_2, \dots, a_{2015}$ be a sequence of positive integers in $[1, 100]$. Call a nonempty contiguous subsequence of this sequence *good* if the product of the integers in it leaves a remainder of 1 when divided by 101. In other words, it is a pair of integers (x, y) such that $1 \le x \le y \le 2015$ and

    $$a_x a_{x+1} \dots a_{y-1} a_y \equiv 1 \pmod{101}.$$

    Find the minimum possible number of good subsequences across all possible (a_i).

    Proposed by Yang Liu.

    Answer. $\boxed{19320}$.

    Solution. Consider the prefix products, i.e. $p_i = a_1 a_2 \dots a_i$. $p_0 = 1$. Note that (x, y) is good iff $p_{x-1} \equiv p_y \pmod{101}$. Let there be s_i prefix products that evaluate to $i \pmod{p}$. Then $\sum_{i=1}^{100} s_i = 2016$. So our answer is

    $$\sum_{i=1}^{100} \binom{s_i}{2} \ge 84\binom{20}{2} + 16\binom{21}{2} = 19320,$$

    by convexity. $\square$

15. A regular 2015-simplex $\mathcal{P}$ has 2016 vertices in 2015-dimensional space such that the distances between every pair of vertices are equal. Let S be the set of points contained inside $\mathcal{P}$ that are closer to its center than any of its vertices. The ratio of the volume of S to the volume of $\mathcal{P}$ is $\frac{m}{n}$, where m and n are relatively prime positive integers. Find the remainder when $m + n$ is divided by 1000.

    Proposed by James Lin.

    Answer. $\boxed{321}$.

    Solution. A point has a $\left(\frac{2015}{4032}\right)^{2015}$ chance of being closer to a particular vertex than the center by barycentric coordinates. Our desired probability is then $1 - 2016\left(\frac{2015}{4032}\right)^{2015}$ since no point can be closer to two vertices than the center. $\square$

16. Given a (nondegenrate) triangle ABC with positive integer angles (in degrees), construct squares BCD_1D_2, ACE_1E_2 outside the triangle. Given that D_1, D_2, E_1, E_2 all lie on a circle, how many ordered triples $(\angle A, \angle B, \angle C)$ are possible?

    Proposed by Yang Liu.

    Answer. $\boxed{223}$.

    Solution. The circumcenter of the 4 points must be the circumcenter of ABC. Now bash using trigonometry to show that either $\angle A = \angle B$, or $\angle A + \angle B = 135°$.

    The first case gives 89 triangles, the second case gives 67 triangles. One triangle is counted twice, namely $\pi/4, \pi/4, \pi/2$. So the answer is $89 + 67 = 156$. □

17. Let $x_1 \ldots, x_{42}$, be real numbers such that $5x_{i+1} - x_i - 3x_ix_{i+1} = 1$ for each $1 \le i \le 42$, with $x_1 = x_{43}$. Find the product of all possible values for $x_1 + x_2 + \cdots + x_{42}$.

    Proposed by Michael Ma.

    Answer. $\boxed{588}$.

    Solution. First we notice that we can rearrange the terms of the condition into $x_{n+1} = \frac{x_n+1}{-3x_n+5}$. So we let $f(x) = \frac{x+1}{-3x+5}$. Now $f(x_n) = x_{n+1}$. So we can see that $f^{(42)}(x_m) = x_m$. So now notice that if we let $A = \begin{matrix} 1 & 1 \\ -3 & 5 \end{matrix}$ that the coefficients of $f^{(n)}(x)$ are the entries of A^n. Now to calculate A^{42} we need to diagnolize A. So diagnolizing A as PBP^{-1} we get $P = \begin{matrix} 1 & 1 \\ 1 & 3 \end{matrix}$ and $B = \begin{matrix} 2 & 0 \\ 0 & 4 \end{matrix}$. So $A^{42} = PB^{42}P^{-1}$. Now calculating by matrix multiplication we can get $A^{42} = \begin{matrix} 3\times2^{99}-2^{199} & 2^{199}-2^{99} \\ 3\times2^{99}-3\times2^{199} & 3\times2^{199}-2^{99} \end{matrix}$. Now subsituting back into $f^{(42)}(x_m) = x_m$ we get that $3x_n^2 - 4x_n + 1 = 0$. So now we conclude that $x_n = 1, \frac{1}{3}$. Also notice that $f(1) = 1$ and $f(\frac{1}{3}) = \frac{1}{3}$. So finishing we see that the two possibilities are 42 and 14. Multiplying we get $42 \times 14 = 588$. □

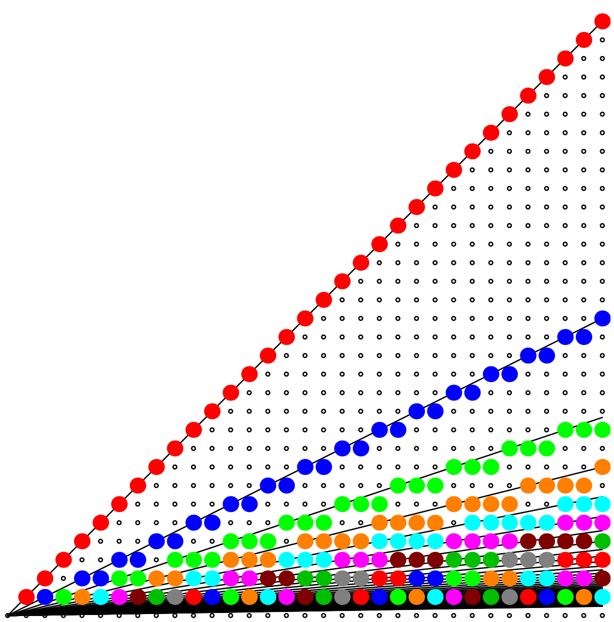
18. Given an integer n, an integer $1 \le a \le n$ is called *n-well* if

    $$\left\lfloor \frac{n}{\lfloor n/a \rfloor} \right\rfloor = a.$$

    Let $f(n)$ be the number of n-well numbers, for each integer $n \ge 1$. Compute $f(1) + f(2) + \ldots + f(9999)$.

    Proposed by Ashwin Sah.

    Answer. $\boxed{1318350}$.

    Solution. □

19. For any set S, let $P(S)$ be its power set, the set of all of its subsets. Over all sets A of 2015 arbitrary finite sets, let N be the maximum possible number of ordered pairs (S,T) such that $S \in P(A), T \in P(P(A))$, $S \in T$, and $S \subseteq T$. (Note that by convention, a set may never contain itself.) Find the remainder when N is divided by 1000.

    Proposed by Ashwin Sah.

    Answer. $\boxed{872}$.

    Solution. We might as well add in trash elements to make $|A| = k$, since this can only increase the amount of ordered pairs in question.

    Now, $T \in P(P(A))$ means that $S \subseteq T \subseteq P(A)$ and $S \in P(A)$ means that $S \subseteq A$. Combining gives $S \subseteq P(A) \cap A$. Let $|P(A) \cap A| = x$.

    Then there are $\binom{x}{i}$ possibilities for S where $|S| = i$. Then T must contain $S, a_1, a_2, \dots, a_i$, where $a_1, \dots, a_i$ are the distinct elements of S, which must be distinct from S itself (since they are elements of A and thus are finitely defined, and since they are also elements of S). Then T has $2^k - i - 1$ other elements that it can include or not, for a total of $\binom{x}{i}2^{2^k-i-1}$ possibilities when $|S| = i$. Vary i to get $\sum_{i=0}^{x}\binom{x}{i}2^{2^k-i-1} = 2^{2^k-1}\left(\frac{3}{2}\right)^x$. Now this is maximized when x is, and $x \le k$ is clear. Furthermore, we can attain $x = k$ at $A = \{\{\}, \{\{\}\}, \{\{\{\}\}\}, \dots\}$, where there are k nested curly braces in the last element, since each element of A is in this case also a subset of A. Then $N_k = 2^{2^k-k-1}3^k$.

    Using $2^{9001} - 9001 - 1 \equiv 6750 \pmod{9000}$ we find that $2^{2^{9001}-9001-1} \equiv \left(2^{2250}\right)^3 \equiv 1 \pmod{9001}$ since $1204^4 \equiv 2 \pmod{9001}$. And $3^{9001} \equiv 3 \pmod{9001}$, so we get the result.

    An easier to understand version of the optimal A is $A = \{\varnothing, \{\varnothing\}, \{\{\varnothing\}\}, \dots\}$, up to k terms, where $\varnothing = \{\}$ is the empty set. $\square$

20. Amandine and Brennon play a turn-based game, with Amadine starting. On their turn, a player must select a positive integer which cannot be represented as a sum of multiples of any of the previously selected numbers. For example, if $3, 5$ have been selected so far, only $1, 2, 4, 7$ are available to be picked; if only 3 has been selected so far, all numbers not divisible by three are eligible. A player loses immediately if they select the integer 1.

    Call a number n *feminist* if $\gcd(n, 6) = 1$ and if Amandine wins if she starts with n. Compute the sum of the *feminist* numbers less than 40.

    Proposed by Ashwin Sah.

    Answer. $\boxed{192}$.

    Solution. By the way, this game is called "Sylver coinage" on Wikipedia. So if this isn't original enough, then feel free to scrap it.

    We claim that the *feminist* numbers are just the prime numbers greater than three. If we can show that each of those primes $p \ge 5$ is a winning position, then we are done - a feminist number n satisfies $\gcd(n, 6) = 1$ and obviously $n > 1$, so n has a prime divisor $q \ge 5$; if $n \ne q$ then after Amandine selects n then Brennon can select q, and it is as if Brennon started with the move q and thus he will win, and it is not a feminist number.

    Suppose Amandine starts with $p \ge 5$, a prime. Then say Brennon does a. Clearly $\gcd(a, p) = 1$, so now there are only finitely many guys that are left to be chosen, and by Chicken McNugget the biggest of these is $ap - a - p > 1$ since $p \ge 5$. We will do a Chomp-like nonconstructive proof.

    Suppose that now Brennon will now win, regardless of what Amandine does. If Amandine does $ap - a - p$, then Brennon can do a winning move b. It is easy to see that $ap - a - p$ is actually a nonnegative combination of a, b, p; then Amandine should have done move b to begin with, and thus win! So actually Amandine wins with some number, we just don't know which.

    Thus the answer is $5 + 7 + 11 + 13 + 17 + 19 + 23 + 29 + 31 + 37 = 192$. $\square$

21. Toner Drum and Celery Hilton are both running for president. A total of 2015 people cast their vote, giving 60% to Toner Drum. Let N be the number of "representative" sets of the 2015 voters that could have been polled to correctly predict the winner of the election (i.e. more people in the set voted for Drum than Hilton). Compute the remainder when N is divided by 2017.

    Proposed by Ashwin Sah.

    Answer. $\boxed{605}$.

    Solution. Suppose m people voted for Celery and n for Toner, where $m < n$. Then the amount of sets of voters that could be chosen with $d > 0$ more people voting for Toner than for Celery is $\binom{m}{0}\binom{n}{d} + \binom{m}{1}\binom{n}{d+1} + \ldots + \binom{m}{m}\binom{n}{m+d}$, where some of the terms at the end might be zero if $m+d > n$. By Vandermonde, this sum is just $\binom{m+n}{m+d}$. Now sum over the possible differences, which are $d = 1, 2, \ldots, n$. We get $\sum_{i=m+1}^{i=m+n} \binom{m+n}{i}$.

    Now $m + n = 2015$, so $\binom{m+n}{i} \equiv \binom{2015}{i} \equiv \frac{i+1}{2016}\binom{2016}{i+1} \equiv \left(\frac{i+1}{2016}\right)(-1)^{i+1} \equiv (i+1)(-1)^i \pmod{2017}$.

    So since $m = \frac{2}{5}(2015) = 806$, we get $-808 + 809 - 810 + \ldots - 2016 \equiv -808 + 604(-1) \equiv -1412 \equiv 605 \pmod{2017}$ □

22. Let $W = \ldots x_{-1}x_0x_1x_2\ldots$ be an infinite periodic word consisting of only the letters a and b. The minimal period of W is 2^{2016}. Say that a word U *appears* in W if there are indices $k \le \ell$ such that $U = x_kx_{k+1}\ldots x_\ell$. A word U is called *special* if Ua, Ub, aU, bU all appear in W. (The empty word is considered special) You are given that there are no special words of length greater than 2015.

    Let N be the minimum possible number of special words. Find the remainder when N is divided by 1000.

    Proposed by Yang Liu.

    Answer. $\boxed{535}$.

    Solution. Firstly, you can show that if a word U appears twice in a period of W, then it is part of a special word. You prove this by appending and pretending letters to both instances of U until the pretended/appended letters don't match between the words. This has to happen by minimal period. By Pigeonhole then, all words of length 2014 appear exactly once in a period of W. So for all words U of length ≤ 2013, aU, bU, Ua, Ub have length at most 2014, so they are the prefix of some word of length 2014 $\implies$ they all appear in W. So our answer is $2^0 + 2^1 + 2^2 + \ldots + 2^{2013} = 2^{2016} - 1$. □

23. Let $p = 2017$, a prime number. Let N be the number of ordered triples (a, b, c) of integers such that $1 \le a, b \le p(p-1)$ and $a^b - b^a = p \cdot c$. Find the remainder when N is divided by 1000000.

    Proposed by Evan Chen and Ashwin Sah.

    Answer. $\boxed{512256}$.

    Solution. ¿¿¿ good = [(a,b) for a in xrange(1,31) for b in xrange(1,31) if (a**b+b**a) ¿¿¿ good [(1, 2), (1, 5), (1, 8), (1, 11), (1, 14), (1, 17), (1, 20), (1, 23), (1, 26), (1, 29), (2, 1), (2, 5), (2, 7), (2, 11), (2, 13), (2, 17), (2, 19), (2, 23), (2, 25), (2, 29), (3, 3), (3, 6), (3, 9), (3, 12), (3, 15), (3, 18), (3, 21), (3, 24), (3, 27), (3, 30), (5, 1), (5, 2), (5, 7), (5, 8), (5, 13), (5, 14), (5, 19), (5, 20), (5, 25), (5, 26), (6, 3), (6, 6), (6, 9), (6, 12), (6, 15), (6, 18), (6, 21), (6, 24), (6, 27), (6, 30), (7, 2), (7, 5), (7, 8), (7, 11), (7, 14), (7, 17), (7, 20), (7, 23), (7, 26), (7, 29), (8, 1), (8, 5), (8, 7), (8, 11), (8, 13), (8, 17), (8, 19), (8, 23), (8, 25), (8, 29), (9, 3), (9, 6), (9, 9), (9, 12), (9, 15), (9, 18), (9, 21), (9, 24), (9, 27), (9, 30), (11, 1), (11, 2), (11, 7), (11, 8), (11, 13), (11, 14), (11, 19), (11, 20), (11, 25), (11, 26), (12, 3), (12, 6), (12, 9), (12, 12), (12, 15), (12, 18), (12, 21), (12, 24), (12, 27), (12, 30), (13, 2), (13, 5), (13, 8), (13, 11), (13, 14), (13, 17), (13, 20), (13, 23), (13, 26), (13, 29), (14, 1), (14, 5), (14, 7), (14, 11), (14, 13), (14, 17), (14, 19), (14, 23), (14, 25), (14, 29), (15, 3), (15, 6), (15, 9), (15, 12), (15, 15), (15, 18), (15, 21), (15, 24), (15, 27), (15, 30), (17, 1), (17, 2), (17, 7), (17, 8), (17, 13), (17, 14), (17, 19), (17, 20), (17, 25), (17, 26), (18, 3), (18, 6), (18, 9), (18, 12), (18, 15), (18, 18), (18, 21), (18, 24), (18, 27), (18, 30), (19, 2), (19, 5), (19, 8), (19, 11), (19, 14), (19, 17), (19, 20), (19, 23), (19, 26), (19, 29), (20, 1), (20, 5), (20, 7), (20, 11), (20, 13), (20, 17), (20, 19), (20, 23), (20, 25), (20, 29), (21, 3), (21, 6), (21, 9), (21, 12), (21, 15), (21, 18), (21, 21), (21, 24), (21, 27), (21, 30), (23, 1), (23, 2), (23, 7), (23, 8), (23, 13), (23, 14), (23, 19), (23, 20), (23, 25), (23, 26), (24, 3), (24, 6), (24, 9), (24, 12), (24, 15), (24, 18), (24, 21), (24, 24), (24, 27), (24, 30), (25, 2), (25, 5), (25, 8), (25, 11), (25, 14), (25, 17), (25, 20), (25, 23), (25, 26), (25, 29), (26, 1), (26, 5), (26, 7), (26, 11), (26, 13), (26, 17), (26, 19), (26, 23), (26, 25), (26, 29), (27, 3), (27, 6), (27, 9), (27, 12), (27, 15), (27, 18), (27, 21), (27, 24), (27, 27), (27, 30), (29, 1), (29, 2), (29, 7), (29, 8), (29, 13), (29, 14), (29, 19), (29, 20), (29, 25), (29, 26), (30, 3), (30, 6), (30, 9), (30, 12), (30, 15), (30, 18), (30, 21), (30, 24), (30, 27), (30, 30)] ¿¿¿ len(good) 250

If 3 divides one of a, b then it divides the other, yielding $10^2 = 100$ valid solutions.

Now if $a \equiv b \equiv 1 \pmod 3$ clearly this fails. If $a \equiv 1 \pmod 3, b \equiv 2 \pmod 3$ then the only extra condition is that $a \equiv 1 \pmod 2$, which gives $5 \cdot 10 = 50$ valid solutions. If $a \equiv 2 \pmod 3, b \equiv 1 \pmod 3$ there are also 50 valid solutions. If $a \equiv b \equiv 2 \pmod 3$ then either a is even and b is odd, for $5^2 = 25$ solutions, or a is odd and b is even, for $5^2 = 25$ solutions.

We get 250 by adding. □

24. Let ABC be an acute triangle with incenter I; ray AI meets the circumcircle Ω of ABC at $M \neq A$. Suppose T lies on line BC such that $\angle MIT = 90^\circ$. Let K be the foot of the altitude from I to $\overline{TM}$. Given that $\sin B = \frac{55}{73}$ and $\sin C = \frac{77}{85}$, and $\frac{BK}{CK} = \frac{m}{n}$ in lowest terms, compute $m + n$.

Proposed by Evan Chen.

Answer. $\boxed{5702}$.

Solution. Let X be the major arc midpoint. Let ray XI meet the circumcircle of BIC (centered at M) again at J. Then $BICJ$ is harmonic, K is the midpoint of IJ and in particular lies on Ω.

Moreover, ray KX is the angle bisector of $\angle BKC$, so if welt L be the intersection of $\overline{IK}$ with $\overline{BC}$ we deduce

$$\frac{BK}{KC} = \frac{BT}{TC} = \left(\frac{BI}{IC}\right)^2 = \frac{\sin^2 C/2}{\sin^2 B/2}.$$

From the gives, we have $\cos B = \frac{48}{73}$, $\cos C = \frac{36}{85}$, and hence the requested ratio is

$$\frac{\frac{1}{2}(1 - \frac{36}{85})}{\frac{1}{2}(1 - \frac{48}{73})} = \frac{49 \cdot 73}{85 \cdot 25} = \frac{3577}{2125}.$$

Hence an answer of $3577 + 2125 = 5702$. □

25. Define $\|A - B\| = (x_A - x_B)^2 + (y_A - y_B)^2$ for every two points $A = (x_A, y_A)$ and $B = (x_B, y_B)$ in the plane. Let S be the set of points (x, y) in the plane for which $x, y \in \{0, 1, \dots, 100\}$. Find the number of functions $f : S \to S$ such that $\|A - B\| \equiv \|f(A) - f(B)\| \pmod{101}$ for any $A, B \in S$.

Proposed by Victor Wang.

Answer. $\boxed{2040200}$.

Solution. We solve this problem with 9001 replaced by an arbitrary prime $p \equiv 1 \pmod 4$.

First translate so that $f(0) = 0$ (we multiply the count by p^2 at the end).

Then considering $(x, 0)$, $(y, 0)$, and (x, y), and noting p is odd yields $f(x) \cdot f(y) \equiv x \cdot y \pmod p$ for all points x, y.

Now fix x, y; then $[f(x + y) - f(x) - f(y)] \cdot f(t) \equiv 0 \pmod p$ for all points t.

Assume for the sake of contradiction that $f(x + y) - f(x) - f(y) = (a, b)$ with at least one of a, b nonzero. (*) Then for all t, we have $(a, b) \cdot f(t) \equiv 0 \pmod p$, or equivalently, $f(t) \in \mathbb{Z} \cdot (-b, a)$. In other words, there exists $g : \mathbb{Z}/p \to \mathbb{Z}/p$ such that $f(t) \equiv (-bg(t), ag(t)) \pmod p$ for all t.

(*) We can also take $t = x + y, x, y$ to get $a^2 + b^2 \equiv 0 \pmod p$, which slightly simplifies the next paragraph.

In particular, we have $u^2+v^2 \equiv |f(u,v)|^2 \equiv g(u,v)^2(b^2+a^2) \pmod p$ for all residues u, v. On the other hand, by Cauchy-Davenport or direct calculations, we know that $\{u^2+v^2\} = \mathbb{Z}/p$. Yet $g(u,v)^2(b^2+a^2)$ is a square times a constant (b^2+a^2), which covers at most $\frac{p+1}{2} < p$ values, contradiction.

Thus $f(x+y) \equiv f(x) + f(y)$ for all x, y. Let $f(1,0) = (A, B)$ and $f(0,1) = (C, D)$ (with $A^2 + B^2 \equiv C^2 + D^2 \equiv 0^2 + 1^2$), so $f(u,v) \equiv (Au + Cv, Bu + Dv)$. By linearity, we just need to check $u^2 + v^2 \equiv (Au + Cv)^2 + (Bu + Dv)^2$, which boils down to $0 \equiv 2(AC + BD)uv$ for all u, v.

(Alternatively, if M denotes the matrix of the linear transformation f over the vector space $(\mathbb{F}_p)^2$, then the dot product condition is equivalent to $x^T(M^TM)y \equiv x^Ty$ for all vectors x, y. Of course, $M = \left[\begin{smallmatrix} A & C \\ B & D \end{smallmatrix}\right]$.)

Therefore our desired answer is simply p^2 times the number of solutions to the system $A^2 + B^2 \equiv C^2 + D^2 \equiv 1$, $AC + BD \equiv 0$ (essentially counting orthogonal matrices over the finite field $\mathbb{F}_p$). We can parameterize $A + iB \equiv \alpha$, $A - iB \equiv \alpha^{-1}$, $C + iD \equiv \beta$, $C - iD \equiv \beta^{-1}$ for nonzero residues α, β. Now $AC + BD \equiv 0$ is equivalent to $(\alpha + \alpha^{-1})(\beta + \beta^{-1}) - (\alpha - \alpha^{-1})(\beta - \beta^{-1}) \equiv 0$, or $\alpha^2 + \beta^2 \equiv 0$. For each (nonzero) α, there are exactly 2 choices for β, so our final answer is $p^2[2(p-1)] = 2p^2(p-1)$. □

26. Let ABC be a triangle with $AB = 72, AC = 98, BC = 110$, and circumcircle Γ, and let M be the midpoint of arc BC not containing A on Γ. Let A' be the reflection of A over BC, and suppose MB meets AC at D, while MC meets AB at E. If MA' meets DE at F, find the distance from F to the center of Γ.

Proposed by Michael Kural.

Answer. $\boxed{231}$.

Solution. Let G be the intersection of BC and AM. By Brokard, EDG is self-polar. We claim that F is the Miquel point of $ABCM$, which is well-known to be inverse of G with respect to O. To prove this, we claim that the Miquel point F' is the inverse of A' with respect to (M) (aka (BIC)), which can be proven with angle chasing. Then clearly the intersection of MA' and DE is this unique point, and $F' = F$. Then we just calculate OG to get OF.

Side note: It was VERY hard to generate a triple of side lengths which makes the final answer rational (it was kind of miraculous that it could be an integer too). Almost all triples which result in a rational solution are either isosceles, right (putting O on BC which makes it trivially rational), or satisfy $BM \parallel AC$ or $CM \parallel AB$. □

27. For integers $0 \le m, n \le 64$, let $\alpha(m,n)$ be the number of nonnegative integers k for which $\lfloor m/2^k \rfloor$ and $\lfloor n/2^k \rfloor$ are both odd integers. Consider a 65×65 matrix M whose (i,j)th entry (for $1 \le i, j \le 65$) is

$$(-1)^{\alpha(i-1,j-1)}.$$

Compute the remainder when $\det M$ is divided by 1000.

Proposed by Evan Chen.

Answer. $\boxed{792}$.

Solution. Let $n = 64$. Show that $(\det M)^2 = 4n^n$, because M^TM is very nice. Hence $\det M = 2n^{n/2}$. □

28. Let N be the number of 2015-tuples of (not necessarily distinct) subsets $(S_1, S_2, \ldots, S_{2015})$ of $\{1, 2, \ldots, 2015\}$ such that the number of permutations σ of $\{1, 2, \ldots, 2015\}$ satisfying $\sigma(i) \in S_i$ for all $1 \le i \le 2015$ is odd. Let k_2, k_3 be the largest integers such that $2^{k_2}|N$ and $3^{k_3}|N$ respectively. Find $k_2 + k_3$.

Proposed by Yang Liu.

Answer. $\boxed{2030612}$.

Solution. Consider each subset as a vector in $\mathbb{F}_2^{2015}$, and write these vectors in a 2015×2015 matrix. Then the number of good permutations is the permanent of this matrix. But permanent is congruent to determinant $\pmod 2$, so we want these 2015 vector to be linearly independent. So the total number of ways is

$$\prod_{i=0}^{2014} (2^{2015} - 2^i).$$

So $k_2 = 2029105, k_3 = 1507$ (by LTE). So $k_2 + k_3 = 2030612$. ☐

29. Given vectors $v_1, \ldots, v_n$ and the string $v_1 v_2 \ldots v_n$, we consider valid expressions formed by inserting $n-1$ sets of balanced parentheses and $n-1$ binary products, such that every product is surrounded by a parentheses and is one of the following forms:

- A "normal product" ab, which takes a pair of scalars and returns a scalar, or takes a scalar and vector (in any order) and returns a vector.
- A "dot product" $a \cdot b$, which takes in two vectors and returns a scalar.
- A "cross product" $a \times b$, which takes in two vectors and returns a vector.

An example of a *valid* expression when $n = 5$ is $(((v_1 \cdot v_2)v_3) \cdot (v_4 \times v_5))$, whose final output is a scalar. An example of an *invalid* expression is $(((v_1 \times (v_2 \times v_3)) \times (v_4 \cdot v_5))$; even though every product is surrounded by parentheses, in the last step one tries to take the cross product of a vector and a scalar.

Denote by T_n the number of valid expressions (with $T_1 = 1$), and let R_n denote the remainder when T_n is divided by 4. Compute $R_1 + R_2 + R_3 + \ldots + R_{1{,}000{,}000}$.

Proposed by Ashwin Sah.

Answer. $\boxed{300}$.

Solution. So let S_n be the amount a scalar result and V_n with a vector result. So $S_1 = 0, V_1 = 1$.

We easily find $S_n = \sum_{k=1}^{n-1} S_k S_{n-k} + \sum_{k=1}^{n-1} V_k V_{n-k}$ and $V_n = 2\sum_{k=1}^{n-1} S_k V_{n-k} + \sum_{k=1}^{n-1} V_k V_{n-k}$.

So if we let $S(x) = S_1 x + S_2 x^2 + \ldots$ then $S(x) = S(x)^2 + V(x)^2, V(x) - x = 2S(x)V(x) + V(x)^2$.

So $V(x) - x = 2S(x)V(x) + S(x) - S(x)^2$, and thus $V(x)(1 - 2S(x)) = x + S(x) - S(x)^2$.

Then $(1 - 2S(x))^2 S(x) = (1 - 2S(x))^2 S(x)^2 + (x + S(x) - S(x)^2)^2$.

Modulo 2 gives $S(x) \equiv x^2 + S(x^4) \pmod 2$ by well-known results, so $S(x) \equiv x^2 + x^8 + x^{32} + x^{128} + \ldots$ is easy to see.

Modulo 2 has $V(x) \equiv x + V(x^2) \pmod 2$, too, so $V(x) \equiv x + x^2 + x^4 + x^8 + \ldots \pmod 2$.

Now we can move on the modulo 4. Notice that stuff within squares can be taken modulo 2.

So $S(x) \equiv (x^2 + x^8 + x^{32} + \ldots)^2 + (x + x^2 + x^4 + x^8 + \ldots)^2$.

And $V(x) = x + V(x)^2 + 2S(x)V(x) \equiv x + (x + x^2 + x^4 + x^8 + \ldots)^2 + 2(x^2 + x^8 + x^{32} + \ldots)(x + x^2 + x^4 + x^8 + \ldots) \pmod 4$.

Add the two to find $S(x) + V(x) \equiv x + 2(x + x^2 + x^4 + x^8 + \ldots)^2 + (x^2 + x^8 + x^{32} + \ldots)^2 + 2(x^2 + x^8 + x^{32} + \ldots)(x + x^2 + x^4 + x^8 + \ldots) \pmod 4$.

Then we can find that it is $1 \pmod 4$ for powers of four, $2 \pmod 4$ for twice powers of four and numbers $n = 2^a + 2^b$ for $a \neq b$ not both even. The rest are divisible by four. (CHECK THIS)

So since $2^0, 2^1, \ldots, 2^{19}$ are the only powers of two at most $1{,}000{,}000$ and since the numbers from $1{,}000{,}001$ to $2^{20} - 1$ are all zero mod 4, we easily find $10(1) + 10(2) + \left(\binom{20}{2} - \binom{10}{2}\right)(2) = 300$ to be our answer. ☐

30. Ryan is learning number theory. He reads about the *Möbius function* $\mu : \mathbb{N} \to \mathbb{Z}$, defined by $\mu(1) = 1$ and

$$\mu(n) = -\sum_{\substack{d|n \\ d\neq n}} \mu(d)$$

for $n > 1$ (here $\mathbb{N}$ is the set of positive integers). However, Ryan doesn't like negative numbers, so he invents his own function: the *dubious function* $\delta : \mathbb{N} \to \mathbb{N}$, defined by the relations $\delta(1) = 1$ and

$$\delta(n) = \sum_{\substack{d|n \\ d\neq n}} \delta(d)$$

for $n > 1$. Help Ryan determine the value of $1000p + q$, where p, q are relatively prime positive integers satisfying

$$\frac{p}{q} = \sum_{k=0}^{\infty} \frac{\delta(15^k)}{15^k}.$$

Proposed by Michael Kural.

Answer. $\boxed{11007}$.

Solution. Outline: let $f(i,j) = \delta(p^i q^j)$. Note that for $i, j > 0$ and not both equal to 1,

$$f(i,j) = 2f(i,j-1) + 2f(i-1,j) - 2f(i-1,j-1)$$

One can easily derive that the corresponding generating function $F(x,y)$ satisfies $F(x,y) = \frac{1}{2}\left(1 + \frac{1}{1-2x-2y+2xy}\right)$. Now consider $F(s, \frac{x}{s})$. This is now a rational function of s with a quadratic in the denominator; we want to find the constant term of this expression. We can do this by considering a partial fraction decomposition and considering the root of the quadratic which tends to 0 as s approaches 0; you can also compute it as a residue at this pole after dividing by s. This yields

$$[s^0]F(s, \frac{x}{s}) = \frac{1}{2}\left(1 + \frac{1}{\sqrt{1 - 12x + 4x^2}}\right)$$

so it sufficies to plug in $x = \frac{1}{15}$. $\square$