OMO Fall 2015
November 6 – 17, 2015

1. Evalute
$$\sqrt{\binom{8}{2}+\binom{9}{2}+\binom{15}{2}+\binom{16}{2}}.$$

2. At a national math contest, students are being housed in single rooms and double rooms; it is known that 75% of the students are housed in double rooms. What percentage of the rooms occupied are double rooms?

3. How many integers between 123 and 321 inclusive have exactly two digits that are 2?

4. Let ω be a circle with diameter AB and center O. We draw a circle ω_A through O and A, and another circle ω_B through O and B; the circles ω_A and ω_B intersect at a point C distinct from O. Assume that all three circles ω, ω_A, ω_B are congruent. If $CO = \sqrt{3}$, what is the perimeter of $\triangle ABC$?

5. Merlin wants to buy a magical box, which happens to be an n-dimensional hypercube with side length 1 cm. The box needs to be large enough to fit his wand, which is 25.6 cm long. What is the minimal possible value of n?

6. Farmer John has a (flexible) fence of length L and two straight walls that intersect at a corner perpendicular to each other. He knows that if he doesn't use any walls, he can enclose a maximum possible area of A_0, and when he uses one of the walls or both walls, he gets a maximum of area of A_1 and A_2 respectively. If $n = \frac{A_1}{A_0} + \frac{A_2}{A_1}$, find $\lfloor 1000n \rfloor$.

7. Define sequence $\{a_n\}$ as following: $a_0 = 0$, $a_1 = 1$, and $a_i = 2a_{i-1} - a_{i-2} + 2$ for all $i \geq 2$. Determine the value of a_{1000}.

8. The two numbers 0 and 1 are initially written in a row on a chalkboard. Every minute thereafter, Denys writes the number $a + b$ between all pairs of consecutive numbers a, b on the board. How many odd numbers will be on the board after 10 such operations?

9. Let $s_1, s_2, \ldots$ be an arithmetic progression of positive integers. Suppose that
$$s_{s_1} = x + 2, \quad s_{s_2} = x^2 + 18, \quad \text{and} \quad s_{s_3} = 2x^2 + 18.$$
Determine the value of x.

10. For any positive integer n, define a function f by
$$f(n) = 2n + 1 - 2^{\lfloor \log_2 n \rfloor + 1}.$$
Let f^m denote the function f applied m times.. Determine the number of integers n between 1 and 65535 inclusive such that $f^n(n) = f^{2015}(2015)$.

11. A trapezoid $ABCD$ lies on the xy-plane. The slopes of lines BC and AD are both $\frac{1}{3}$, and the slope of line AB is $-\frac{2}{3}$. Given that $AB = CD$ and $BC < AD$, the absolute value of the slope of line CD can be expressed as $\frac{m}{n}$, where m, n are two relatively prime positive integers. Find $100m + n$.

12. Let a, b, c be the distinct roots of the polynomial $P(x) = x^3 - 10x^2 + x - 2015$. The cubic polynomial $Q(x)$ is monic and has distinct roots $bc - a^2$, $ca - b^2$, $ab - c^2$. What is the sum of the coefficients of Q?

13. You live in an economy where all coins are of value $1/k$ for some positive integer k (i.e. $1, 1/2, 1/3, \ldots$). You just recently bought a coin exchanging machine, called the *Cape Town Machine*. For any integer $n > 1$, this machine can take in n of your coins of the same value, and return a coin of value equal to the sum of values of those coins (provided the coin returned is part of the economy). Given that the product of coins values that you have is 2015^{-1000}, what is the maximum number of times you can use the machine over all possible starting sets of coins?

14. Let $a_1, a_2, \ldots, a_{2015}$ be a sequence of positive integers in $[1, 100]$. Call a nonempty contiguous subsequence of this sequence *good* if the product of the integers in it leaves a remainder of 1 when divided by 101. In other words, it is a pair of integers (x, y) such that $1 \le x \le y \le 2015$ and

$$a_x a_{x+1} \ldots a_{y-1} a_y \equiv 1 \pmod{101}.$$

Find the minimum possible number of good subsequences across all possible (a_i).

15. A regular 2015-simplex $\mathcal{P}$ has 2016 vertices in 2015-dimensional space such that the distances between every pair of vertices are equal. Let S be the set of points contained inside $\mathcal{P}$ that are closer to its center than any of its vertices. The ratio of the volume of S to the volume of $\mathcal{P}$ is $\frac{m}{n}$, where m and n are relatively prime positive integers. Find the remainder when $m + n$ is divided by 1000.

16. Given a (nondegenrate) triangle ABC with positive integer angles (in degrees), construct squares BCD_1D_2, ACE_1E_2 outside the triangle. Given that D_1, D_2, E_1, E_2 all lie on a circle, how many ordered triples $(\angle A, \angle B, \angle C)$ are possible?

17. Let $x_1 \ldots, x_{42}$, be real numbers such that $5x_{i+1} - x_i - 3x_i x_{i+1} = 1$ for each $1 \le i \le 42$, with $x_1 = x_{43}$. Find the product of all possible values for $x_1 + x_2 + \cdots + x_{42}$.

18. Given an integer n, an integer $1 \le a \le n$ is called *n-well* if

$$\left\lfloor \frac{n}{\lfloor n/a \rfloor} \right\rfloor = a.$$

Let $f(n)$ be the number of n-well numbers, for each integer $n \ge 1$. Compute $f(1)+f(2)+\ldots+f(9999)$.

19. For any set S, let $P(S)$ be its power set, the set of all of its subsets. Over all sets A of 2015 arbitrary finite sets, let N be the maximum possible number of ordered pairs (S, T) such that $S \in P(A), T \in P(P(A))$, $S \in T$, and $S \subseteq T$. (Note that by convention, a set may never contain itself.) Find the remainder when N is divided by 1000.

20. Amandine and Brennon play a turn-based game, with Amadine starting. On their turn, a player must select a positive integer which cannot be represented as a sum of multiples of any of the previously selected numbers. For example, if $3, 5$ have been selected so far, only $1, 2, 4, 7$ are available to be picked; if only 3 has been selected so far, all numbers not divisible by three are eligible. A player loses immediately if they select the integer 1.

Call a number n *feminist* if $\gcd(n, 6) = 1$ and if Amandine wins if she starts with n. Compute the sum of the *feminist* numbers less than 40.

21. Toner Drum and Celery Hilton are both running for president. A total of 2015 people cast their vote, giving 60% to Toner Drum. Let N be the number of "representative" sets of the 2015 voters that could have been polled to correctly predict the winner of the election (i.e. more people in the set voted for Drum than Hilton). Compute the remainder when N is divided by 2017.

22. Let $W = \ldots x_{-1}x_0x_1x_2 \ldots$ be an infinite periodic word consisting of only the letters a and b. The minimal period of W is 2^{2016}. Say that a word U *appears* in W if there are indices $k \le \ell$ such that $U = x_k x_{k+1} \ldots x_\ell$. A word U is called *special* if Ua, Ub, aU, bU all appear in W. (The empty word is considered special) You are given that there are no special words of length greater than 2015.

Let N be the minimum possible number of special words. Find the remainder when N is divided by 1000.

23. Let $p = 2017$, a prime number. Let N be the number of ordered triples (a, b, c) of integers such that $1 \le a, b \le p(p-1)$ and $a^b - b^a = p \cdot c$. Find the remainder when N is divided by 1000000.

24. Let ABC be an acute triangle with incenter I; ray AI meets the circumcircle Ω of ABC at $M \neq A$. Suppose T lies on line BC such that $\angle MIT = 90^\circ$. Let K be the foot of the altitude from I to $\overline{TM}$. Given that $\sin B = \frac{55}{73}$ and $\sin C = \frac{77}{85}$, and $\frac{BK}{CK} = \frac{m}{n}$ in lowest terms, compute $m + n$.

25. Define $\|A - B\| = (x_A - x_B)^2 + (y_A - y_B)^2$ for every two points $A = (x_A, y_A)$ and $B = (x_B, y_B)$ in the plane. Let S be the set of points (x, y) in the plane for which $x, y \in \{0, 1, \dots, 100\}$. Find the number of functions $f : S \to S$ such that $\|A - B\| \equiv \|f(A) - f(B)\| \pmod{101}$ for any $A, B \in S$.

26. Let ABC be a triangle with $AB = 72, AC = 98, BC = 110$, and circumcircle Γ, and let M be the midpoint of arc BC not containing A on Γ. Let A' be the reflection of A over BC, and suppose MB meets AC at D, while MC meets AB at E. If MA' meets DE at F, find the distance from F to the center of Γ.

27. For integers $0 \le m, n \le 64$, let $\alpha(m, n)$ be the number of nonnegative integers k for which $\lfloor m/2^k \rfloor$ and $\lfloor n/2^k \rfloor$ are both odd integers. Consider a 65×65 matrix M whose (i, j)th entry (for $1 \le i, j \le 65$) is

$$(-1)^{\alpha(i-1, j-1)}.$$

Compute the remainder when $\det M$ is divided by 1000.

28. Let N be the number of 2015-tuples of (not necessarily distinct) subsets $(S_1, S_2, \dots, S_{2015})$ of $\{1, 2, \dots, 2015\}$ such that the number of permutations σ of $\{1, 2, \dots, 2015\}$ satisfying $\sigma(i) \in S_i$ for all $1 \le i \le 2015$ is odd. Let k_2, k_3 be the largest integers such that $2^{k_2}|N$ and $3^{k_3}|N$ respectively. Find $k_2 + k_3$.

29. Given vectors $v_1, \dots, v_n$ and the string $v_1 v_2 \dots v_n$, we consider valid expressions formed by inserting $n - 1$ sets of balanced parentheses and $n - 1$ binary products, such that every product is surrounded by a parentheses and is one of the following forms:

- A "normal product" ab, which takes a pair of scalars and returns a scalar, or takes a scalar and vector (in any order) and returns a vector.
- A "dot product" $a \cdot b$, which takes in two vectors and returns a scalar.
- A "cross product" $a \times b$, which takes in two vectors and returns a vector.

An example of a *valid* expression when $n = 5$ is $(((v_1 \cdot v_2)v_3) \cdot (v_4 \times v_5))$, whose final output is a scalar. An example of an *invalid* expression is $(((v_1 \times (v_2 \times v_3)) \times (v_4 \cdot v_5))$; even though every product is surrounded by parentheses, in the last step one tries to take the cross product of a vector and a scalar.

Denote by T_n the number of valid expressions (with $T_1 = 1$), and let R_n denote the remainder when T_n is divided by 4. Compute $R_1 + R_2 + R_3 + \ldots + R_{1,000,000}$.

30. Ryan is learning number theory. He reads about the *Möbius function* $\mu : \mathbb{N} \to \mathbb{Z}$, defined by $\mu(1) = 1$ and

$$\mu(n) = -\sum_{\substack{d|n \\ d \neq n}} \mu(d)$$

for $n > 1$ (here $\mathbb{N}$ is the set of positive integers). However, Ryan doesn't like negative numbers, so he invents his own function: the *dubious function* $\delta : \mathbb{N} \to \mathbb{N}$, defined by the relations $\delta(1) = 1$ and

$$\delta(n) = \sum_{\substack{d|n \\ d \neq n}} \delta(d)$$

for $n > 1$. Help Ryan determine the value of $1000p + q$, where p, q are relatively prime positive integers satisfying

$$\frac{p}{q} = \sum_{k=0}^{\infty} \frac{\delta(15^k)}{15^k}.$$