NYCMT 2024-2025 Homework #3 Solutions

NYCMT

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Problem 1. Let a_1, b_1, c_1 be positive integers such that $a_1^2 + b_1^2 = c_1^2$, and let a_2, b_2, c_2 be positive integers such that $a_2^2 + b_2^2 = c_2^2$. Characterize all cases where $(a_1 + a_2)^2 + (b_1 + b_2)^2 = (c_1 + c_2)^2$.

Answer. $\boxed{(a_2, b_2) \text{ is a scalar multiple of } (a_1, b_1)}$

Solution. Consider a triangle in the plane that has vertices $(0, 0)$, (a_1, b_1), and $(a_1 + a_2, b_1 + b_2)$. Its side lengths are $\sqrt{a_1^2 + b_1^2} = c_1$, $\sqrt{a_2^2 + b_2^2} = c_2$, and

$$\sqrt{(a_1 + a_2)^2 + (b_1 + b_2)^2} = c_1 + c_2.$$

By the Triangle Inequality, this is only possible when the triangle is degenerate and the vertices are collinear. Thus, it must be the case that

$$\frac{a_1}{a_2} = \frac{b_1}{b_2},$$

so (a_2, b_2) is a (not necessarily integer) scalar multiple of (a_1, b_1). $\square$

Problem 2. Two acute angles a and b satisfy $\sin^2(a)+\sin^2(b) = \sin(a+b)$. Prove that $a+b = \frac{\pi}{2}$.

Solution. Let $c = \frac{\pi}{2} - b$. We want to show that $a = c$, given

$$\sin^2(a) + \sin^2(b) = \sin(a)\cos(b) + \cos(a)\sin(b),$$

or equivalently,

$$\sin^2(a) + \cos^2(c) = \sin(a)\sin(c) + \cos(a)\cos(c).$$

Rearranging, we get

$$\sin^2(a) - \sin(a)\sin(c) = \cos(a)\cos(c) - \cos^2(c),$$

and factoring both sides, this simplifies to

$$\sin(a) \cdot (\sin(a) - \sin(c)) = \cos(c) \cdot (\cos(a) - \cos(c)).$$

Since a and b are acute, we know that $a, b \in [0, \frac{\pi}{2}]$. Thus, $\sin(a)$ and $\cos(c)$ are both positive. Furthermore, notice that for $x \in [0, \frac{\pi}{2}]$, $\sin x$ is strictly increasing and $\cos x$ is strictly decreasing. Thus, if $a \neq c$, then $\sin(a) - \sin(c)$ and $\cos(a) - \cos(c)$ must be opposite signs, so there are no solutions when $a \neq c$.

When $a = c$, we get $\sin^2(a) + \cos^2(a) = \sin(a)\sin(a) + \cos(c)\cos(c)$, which is clearly true. $\square$

Problem 3. Prove that for arbitrary reals $x_1, x_2, \dots, x_n \in [0,1]$, we have that $(x_1 + x_2 + \cdots + x_n + 1)^2 \geq 4(x_1^2 + x_2^2 + \cdots + x_n^2)$.

Solution. We prove the result through induction on n.
Base Case: $n = 1$
We want to show that for $x_1 \in [0,1]$,

$$(x_1 + 1)^2 \geq 4x_1^2.$$

Notice that for $x_1 \in [0,1]$,

$$(3x_1 + 1)(x_1 - 1) \leq 0.$$

Expanding this and adding x_1^2 on both sides, we get

$$4x_1^2 - 2x_1 - 1 \leq x_1^2.$$

Adding $2x_1 + 1$ to both sides produces the desired inequality. The base case thus holds.

Inductive Hypothesis: We assume the claim is true for $n = k$, where k is a positive integer. That is, we assume that given $x_1, x_2, \dots, x_k \in [0,1]$, we have that

$$(x_1 + x_2 + \cdots + x_k + 1)^2 \geq 4(x_1^2 + x_2^2 + \cdots + x_k^2).$$

Inductive Step: We want to show that the claim is true for $n = k+1$. In other words, we want to show that given $x_1, x_2, \dots, x_{k+1} \in [0,1]$, we have that

$$(x_1 + x_2 + \cdots + x_{k+1} + 1)^2 \geq 4(x_1^2 + x_2^2 + \cdots + x_{k+1}^2).$$

WLOG let $x_1 \geq x_2 \geq \cdots \geq x_{k+1}$. We claim that

$$x_1 + \cdots + x_k + 1 \geq \frac{3}{2}x_{k+1}.$$

If $x_{k+1} \leq \frac{2}{3}$, then $\frac{3}{2}x_{k+1} \leq 1$. Since all $x_i \in [0,1]$,

$$x_1 + \cdots + x_k + 1 \geq 1 \geq \frac{3}{2}x_{k+1},$$

as desired. If $x_{k+1} > \frac{2}{3}$, then because $x_1 \geq x_2 \geq \cdots \geq x_{k+1}$ and k is a positive integer (so $k \geq 1$), we have that

$$x_1 + \cdots + x_k + 1 > \frac{2}{3} + 1 > \frac{3}{2} \geq \frac{3}{2}x_{k+1},$$

since we're given that $x_{k+1} \in [0,1]$. This proves the claim. We can take this claim and multiply both sides by $2x_{k+1}$ and then add x_{k+1}^2 on both sides, producing the following inequality:

$$2x_{k+1}(x_1 + \cdots + x_k + 1) + x_{k+1}^2 \geq 4x_{k+1}^2.$$

By the inductive hypothesis,

$$(x_1 + \cdots + x_k + 1)^2 \geq 4(x_1^2 + \cdots + x_k^2).$$

Adding the two inequalities, we get that

$$(x_1 + \cdots + x_k + 1)^2 + 2x_{k+1}(x_1 + \cdots + x_k + 1) + x_{k+1}^2 \geq 4(x_1^2 + \cdots + x_k^2) + 4x_{k+1}^2,$$

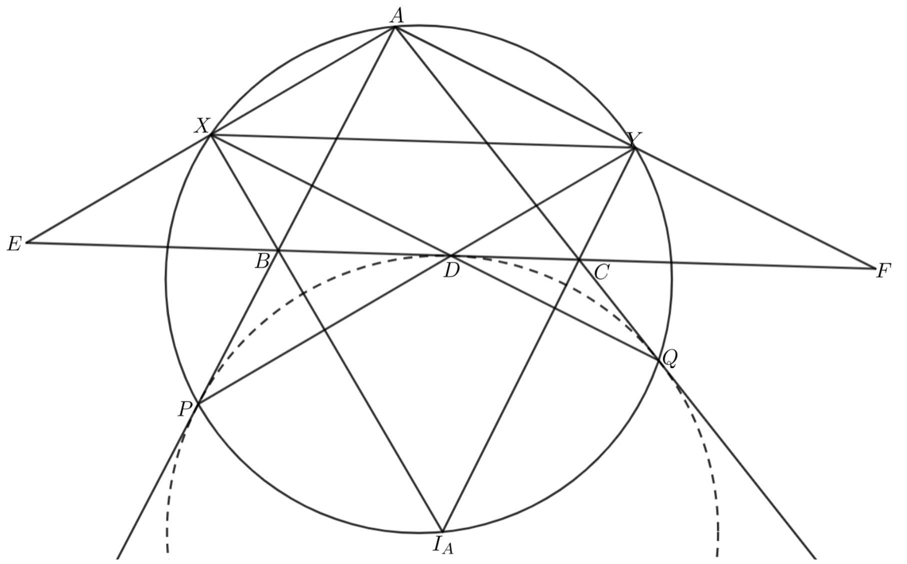
so

$$((x_1 + \cdots + x_k + 1) + x_{k+1})^2 \geq 4(x_1^2 + \cdots + x_k^2 + x_{k+1}^2),$$

as desired. This completes the proof by induction on n. $\square$

Problem 4. Let ABC be a triangle with perimeter 1. The A-excircle touches AB and AC at P and Q. The line passing through the midpoints of AB and AC meets the circumcircle of APQ at two points X and Y. Find the length of XY.

Answer. $\boxed{1/2}$



Solution 1. Let I_A be the A-excenter of $\triangle ABC$. Then since the excircle is tangent to lines $\overleftrightarrow{AB}$ and $\overleftrightarrow{AC}$, we have that

$$\angle API_A = \angle AQI_A = 90^\circ,$$

so I_A lies on the same circle as A, P, Q, X, Y. Let D be the point where the A-excircle of $\triangle ABC$ touches $\overline{BC}$.

Lemma. Lines $\overleftrightarrow{QD}$ and $\overleftrightarrow{I_AB}$ intersect on (APQ).
Let lines $\overleftrightarrow{QD}$ and $\overleftrightarrow{I_AB}$ intersect at X'. Note that

$$\angle QX'I_A = \angle DX'B = 180^\circ - \angle X'BD - \angle X'DB = \angle CBI_A - \angle CDQ.$$

Because $\overline{BI_A}$ bisects $\angle CBP$, we have that

$$\angle CBI_A = \frac{1}{2}(180^\circ - \angle B) = \frac{1}{2}(\angle A + \angle C),$$

and by the Two Tangent Theorem, $CD = CQ$, so

$$\angle CDQ = \frac{1}{2}(180^\circ - \angle DCQ) = \frac{1}{2}\angle C.$$

Thus,

$$\angle QX'I_A = \frac{1}{2}(\angle A + \angle C) - \frac{1}{2}\angle C = \frac{1}{2}\angle A = \angle QAI_A,$$

since $\overline{AI_A}$ is a bisector of $\angle A$. This implies the desired result.

Lemma. Let lines $\overleftrightarrow{AX'}$ and $\overleftrightarrow{BC}$ intersect at E. Then X' is the midpoint of

$\overline{AE}$.
First, notice that since X' lies on (APQ),

$$\angle AX'B = \angle AX'I_A = 90^\circ.$$

Furthermore, we know that $\overline{BX'}$ bisects $\angle ABE$, since X', B, and I_A are collinear. Thus, $\triangle ABE$ must be isosceles with $AB = BE$, implying the result.

We can similarly let Y' be the intersection of $\overleftrightarrow{PD}$ and $\overleftrightarrow{I_AC}$, and let F be the intersection of $\overleftrightarrow{BC}$ and $\overleftrightarrow{AY'}$. By symmetry, Y' also lies on (APQ) and is the midpoint of $\overline{AF}$. Then $\overleftrightarrow{X'Y'}$ must be a midline of $\triangle AEF$, so it must be a midline of $\triangle ABC$ as well. Since we also know that X' and Y' line on (APQ), this implies that $X = X'$ and $Y = Y'$.

Now notice that since BDI_AP is a cyclic quadrilateral,

$$\angle XYP = \angle XI_AP = \angle BI_AP = \angle BDP = \angle BPD = \angle APY,$$

implying that $AXPY$ is an isosceles trapezoid. Thus, $XY = AP$. Now notice that $AP = AQ$ by the Two Tangent Theorem, and that

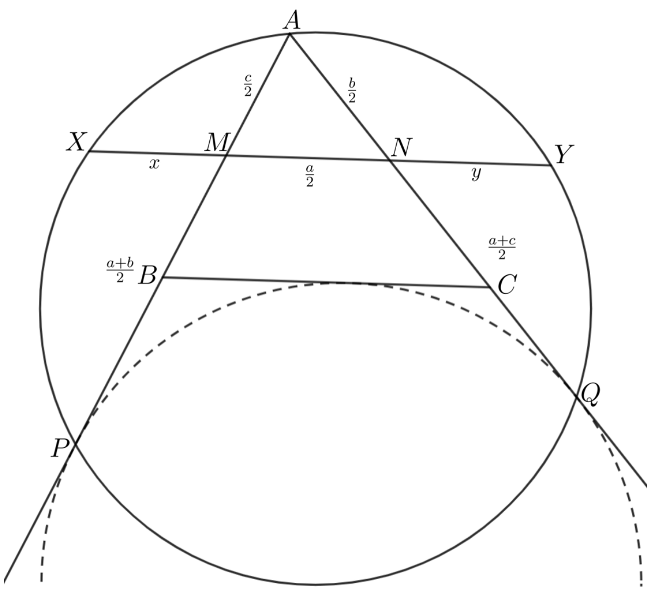
$$AP + AQ = AB + BP + AC + CQ = AB + BD + AC + CD = 1,$$

since the perimeter of $\triangle ABC$ is 1. Thus,

$$XY = AP = \boxed{\frac{1}{2}},$$

as desired. □

Solution 2. (Power of a Point Bash) Let $a = BC$, $b = AC$, and $c = AB$. Further-



more, let M and N be the midpoints of $\overline{AB}$ and $\overline{AC}$, respectively. Then $AM = \frac{c}{2}$, $AN = \frac{b}{2}$, and $MN = \frac{a}{2}$. As shown at the end of the first solution, $AP = AQ$ and both have length equal to half the perimeter of $\triangle ABC$. Thus,

$$MP = \frac{a+b}{2} \text{ and } NQ = \frac{a+c}{2}.$$

Now let $x = XM$ and $y = YN$. Using Power of a Point, we know that

$$XM \cdot MY = AM \cdot MP \text{ and } YN \cdot NX = AN \cdot NQ,$$

so we can set up the following system of equations:

$$\begin{cases} x\left(\frac{a}{2}+y\right) = \frac{c}{2}\left(\frac{a+b}{2}\right) \\ y\left(\frac{a}{2}+x\right) = \frac{b}{2}\left(\frac{a+c}{2}\right). \end{cases}$$

Expanding both equations, we obtain the following system:

$$\begin{cases} xy + \frac{xa}{2} = \frac{ca+cb}{4} \\ xy + \frac{ya}{2} = \frac{ab+cb}{4}. \end{cases}$$

Subtracting these two equations, we get that

$$\frac{xa-ya}{2} = \frac{ca-ab}{4},$$

so

$$\frac{a}{2}(x-y) = \frac{a}{2}\left(\frac{c}{2}-\frac{b}{2}\right).$$

Since we know a is nonzero, we can divide both sides by $\frac{a}{2}$ to obtain the equation

$$x - y = \frac{c}{2} - \frac{b}{2}.$$

We can substitute $x = \frac{c}{2} - \frac{b}{2} + y$ into the second equation in the original system to get

$$y\left(\frac{a}{2}+\frac{c}{2}-\frac{b}{2}+y\right) = \frac{b}{2}\left(\frac{a+c}{2}\right).$$

Expanding and moving all the terms to one side, we get the equation

$$y^2 + \left(\frac{a}{2}+\frac{c}{2}-\frac{b}{2}\right)y - \frac{b}{2}\left(\frac{a+c}{2}\right) = 0.$$

Notice that this factors into

$$\left(y-\frac{b}{2}\right)\left(y+\frac{a}{2}+\frac{c}{2}\right) = 0.$$

Thus, either $y = \frac{b}{2}$ or $y = -\frac{a}{2} - \frac{c}{2}$, but the latter is not possible because a, c, and y are all positive. So $y = \frac{b}{2}$, and since $x - y = \frac{c}{2} - \frac{b}{2}$, we find that $x = \frac{c}{2}$. Thus,

$$XY = x + \frac{a}{2} + y = \frac{a+b+c}{2} = \boxed{\frac{1}{2}},$$

as desired. $\square$

Problem 5. Find all non-negative integer solutions (w, x, y, z) with $w \leq x \leq y \leq z$ which satisfy $w^2 + x^2 + y^2 + z^2 = 2^{2004}$.

Answer. $\boxed{(2^{1001}, 2^{1001}, 2^{1001}, 2^{1001}) \text{ and } (0, 0, 0, 2^{1002})}$

Solution. Note that the only possible squares mod 8 are 0, 1, and 4, where 1 is achieved by any odd square, and 0 and 4 are achieved by even squares. Since

$$w^2 + x^2 + y^2 + z^2 \equiv 0 \pmod 8,$$

which is impossible if any of the squares are $1 \pmod 8$, it must be the case that w^2, x^2, y^2, and z^2 are all even squares, and thus divisible by 4. We can then let $w = 2w_1$, $x = 2x_1$, $y = 2y_1$, and $z = 2z_1$, so

$$w_1^2 + x_1^2 + y_1^2 + z_1^2 = 2^{2002}.$$

Since 2^{2002} is still divisible by 8, we see that w_1, x_1, y_1, and z_1 are also all even, and letting $w_1 = 2w_2$, $x_1 = 2x_2$, $y_1 = 2y_2$, and $z_1 = 2z_2$ gives

$$w_2^2 + x_2^2 + y_2^2 + z_2^2 = 2^{2000}.$$

We may descend until

$$w_{1001}^2 + x_{1001}^2 + y_{1001}^2 + z_{1001}^2 = 2^2 = 4,$$

by which point it is clear that the only solutions are $w_{1001} = x_{1001} = y_{1001} = z_{1001} = 1$ and $w_{1001} = x_{1001} = y_{1001} = 0, z_{1001} = 2$. Working backwards, since $w = w_{1001} \cdot 2^{1001}$ (and symmetrically for the other three variables), our two solutions are $\boxed{(2^{1001}, 2^{1001}, 2^{1001}, 2^{1001}) \text{ and } (0, 0, 0, 2^{1002})}$ as desired. $\square$

Problem 6. Alice and Bob play a game of popping bubble wraps. There is a square sheet of 6 by 6 bubbles, and each person takes turns popping however many bubbles they want in one specific row only, with Alice going first. Whichever player pops the last bubble loses. Who wins with optimal play?

Answer. $\boxed{\text{Bob}}$

Solution. Due to the symmetric nature of the rows, Bob employs a copycat strategy with some nuances:

1. For any number of bubbles Alice removes on her turn in one row, Bob removes the same number of bubbles from a different row. These two rows are forever linked in symmetry, where if Alice removes bubbles from one row, Bob will remove the same number from the other. Hence, the number of bubbles in each pair of rows stays the same.

2. Bob will lose if he continues employing this copycat strategy until the very end, as he is bound to take the last bubble. However, Bob can change the outcome of the game towards the very end, when every row has one or no bubbles left and the parity is set. As the strategy in (1) is played out, the number of bubbles must invariably go down, until there exists only one row that contains more than one bubble. This is guaranteed by the strategy in (1), since the parity of rows with a bubble count greater than 1 is always even during Alice's turn. Assessing the remaining bubbles left, Bob can remove all the bubbles in the last row or all but one, fixing the outcome of the remainder of the game by parity.

As an example, consider the following game. Let the ordered 6-tuple $(a_1, a_2, a_3, \cdots, a_6)$ denote the number of bubbles left in rows $1, 2, 3, \cdots, 6$, respectively.

□

Alice's Move	Bob's Move
(0, 6, 6, 6, 6, 6)	(0, 0, 6, 6, 6, 6)
(0, 0, 6, 2, 6, 6)	(0, 0, 2, 2, 6, 6)
(0, 0, 2, 2, 3, 6)	(0, 0, 2, 2, 3, 3)
(0, 0, 2, 2, 1, 3)	(0, 0, 2, 2, 1, 1)
(0, 0, 0, 2, 1, 1)	(0, 0, 0, 1, 1, 1)
(0, 0, 0, 0, 1, 1)	(0, 0, 0, 0, 0, 1)

Table 1: Bob Wins