NYCMT 2024-2025 Homework #3

NYCMT

October 18 - November 8, 2024

These problems are due November 8. Please solve as many problems as you can, and write up solutions (**not just answers!**) to the ones you solve. Write down any progress you make on problems you don't solve. Please write solutions for different questions on separate pages. Make sure to write your name on each page and page numbers per problem.

If you're not going to be present on November 8, you can scan your solutions and email them to ashleyzhu111@gmail.com, sjschool26@gmail.com, and stevenytlou@gmail.com. If you will be there, just hand in your responses on paper. If you have any questions, feel free to ask one of us on Discord or using one of the above emails.

Problems are NOT difficulty-ordered, so you should read and try all of them. Enjoy!

Problem 1. Let a_1, b_1, c_1 be positive integers such that $a_1^2 + b_1^2 = c_1^2$, and let a_2, b_2, c_2 be positive integers such that $a_2^2 + b_2^2 = c_2^2$. Characterize all cases where $(a_1 + a_2)^2 + (b_1 + b_2)^2 = (c_1 + c_2)^2$.

Problem 2. Two acute angles a and b satisfy $\sin^2(a) + \sin^2(b) = \sin(a+b)$. Prove that $a + b = \frac{\pi}{2}$.

Problem 3. Prove that for arbitrary reals $x_1, x_2, ..., x_n \in [0, 1]$, we have that $(x_1 + x_2 + ... + x_n + 1)^2 \geq 4(x_1^2 + x_2^2 + ... + x_n^2)$.

Problem 4. Let ABC be a triangle with perimeter 1. The A-excircle touches AB and AC at P and Q. The line passing through the midpoints of AB and AC meets the circumcircle of APQ at two points X and Y. Find the length of XY.

Problem 5. Find all non-negative integer solutions (w, x, y, z) with $w \leq x \leq y \leq z$ which satisfy $w^2 + x^2 + y^2 + z^2 = 2^{2004}$.

Problem 6. Alice and Bob play a game of popping bubble wraps. There is a square sheet of 6 by 6 bubbles, and each person takes turns popping however many bubbles they want in one specific row only, with Alice going first. Whichever player pops the last bubble loses. Who wins with optimal play?