# NYCMT 2024-2025 Homework #1 Solutions

NYCMT

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**Problem 1.** Consider complex polynomials $P(x) = x^n + a_1x^{n-1} + \cdots + a_n$ with the zeroes $x_1, x_2, \cdots, x_n$ and $Q(x) = x^n + b_1x^{n-1} + \cdots + b_n$ with the zeroes $x_1^2, x_2^2, \cdots, x_n^2$. Prove that if $a_1 + a_3 + a_5 + \cdots$ and $a_2 + a_4 + a_6 + \cdots$ are real numbers, then $b_1 + b_2 + b_3 + \cdots + b_n$ is also real.

*Solution.* Let $A = a_1 + a_3 + a_5 + \cdots$ and $B = a_2 + a_4 + a_6 + \cdots$. Since

$$b_1 + b_2 + b_3 + \cdots + b_n = Q(1) - 1,$$

it is sufficient to show that $Q(1)$ is real. Note that

$$\begin{aligned} Q(1) &= \prod_{i=1}^{n}(1 - x_i^2) \\ &= \prod_{i=1}^{n}(1 - x_i)(1 + x_i) \\ &= (-1)^n \prod_{i=1}^{n}(1 - x_i)(-1 - x_i) \\ &= (-1)^n P(1)P(-1). \end{aligned}$$

We know that

$$\begin{aligned} P(1) &= 1 + a_1 + a_2 + a_3 + \dots \\ &= 1 + A + B, \end{aligned}$$

and

$$\begin{aligned} (-1)^n P(-1) &= (-1)^n \left((-1)^n + (-1)^{n-1}a_1 + (-1)^{n-2}a_2 + (-1)^{n-3}a_3 + \cdots\right) \\ &= 1 - a_1 + a_2 - a_3 + \dots \\ &= 1 - A + B. \end{aligned}$$

Thus,

$$Q(1) = (-1)^n P(1)P(-1) = (-1)^n(1 + A + B)(1 - A + B).$$

Since $A$ and $B$ are real, $Q(1)$ is real, as desired. □

**Problem 2.** Ashley and Sophia are playing a game. They take turns flipping a coin, with Ashley going first, and keep track of the total number of heads they have flipped. Whoever reaches 2 heads flipped first wins. What is the probability that Sophia wins? (Note: The two heads need not be consecutive.)

**Answer.** $\boxed{\frac{11}{27}}$

*Solution.* Let $p$ be the probability that Sophia wins. Then notice that if Ashley's first flip is tails, Sophia has a $1-p$ probability of winning, since it is equivalent to playing the same game with the roles reversed. Thus, we can write the equation

$$p = \frac{1}{2}(1-p) + \frac{1}{2} \cdot P(\text{Sophia wins} \mid \text{Ashley's first flip is heads}).$$

To find the probability that Sophia wins given that Ashley's first flip is heads, we find the probability this happens in $n$ moves and add them up for all possible values of $n$. (Note that $n \geq 2$, since it's impossible to win in less than 2 moves.)

Ashley's first coin flip is given to be a head, and the next $n-1$ coin flips must be tails so that she does not win. There is a $\frac{1}{2^{n-1}}$ probability that this sequence occurs.
On the other hand, Sophia's $n$th coin flip must be a head (since she wins on the $n$th move), and exactly one of her first $n-1$ coin flips is a head; the rest are tails. Then there is a $\frac{n-1}{2^n}$ probability that this sequence occurs.

Thus, the probability Sophia wins on the $n$th move given that Ashley's first flip is heads is

$$\frac{n-1}{2^{2n-1}},$$

so the probability that Sophia wins given that Ashley's first flip is heads is

$$\sum_{n\geq 2} \frac{n-1}{2^{2n-1}} = \frac{1}{2}\sum_{n\geq 1} \frac{n}{4^n}.$$

We then use the identity

$$\frac{1}{(1-x)^2} = \sum_{n\geq 0}(n+1)x^n = 1 + 2x + 3x^2 + 4x^3 + \dots$$

(when $|x| < 1$ and the sequence converges), which either be proved by squaring the geometric series expansion of $\frac{1}{1-x}$ and using stars and bars, or by differentiating that expression. We get

$$\frac{1}{2}\sum_{n\geq 1}\frac{n}{4^n} = \frac{\frac{1}{4}}{2 \cdot \left(1 - \frac{1}{4}\right)^2} = \frac{2}{9}.$$

Thus, we have that

$$p = \frac{1}{2}(1-p) + \frac{1}{2} \cdot \frac{2}{9},$$

and we can solve to get $\boxed{p = \frac{11}{27}}$. □

**Problem 3.** Let $N$ be the number of ordered triples of positive integers $(a, b, c)$ are such that $\operatorname{lcm}(a, b, c) = 20!$ and $\gcd(a, b, c) = 1$. Find the number of divisors of $N$.

**Answer.** $\boxed{176}$

*Solution.* Let $a = 2^{a_1} \cdot 3^{a_2} \cdot 5^{a_3} \cdot 7^{a_4} \cdot 11^{a_5} \cdot 13^{a_6} \cdot 17^{a_7} \cdot 19^{a_8}$, and denote $b$ and $c$ similarly. Note that for $1 \le i \le 8$, we have

$$\min(a_i, b_i, c_i) = 0 \text{ and } \max(a_i, b_i, c_i) = n,$$

for some $n$. We now want to find $f(n)$, which we define to be the number of ways to choose an ordered triple $(a_i, b_i, c_i)$ satisfying these properties. To do this, we use the Principle of Inclusion-Exclusion.

There are $(n+1)^3$ total ordered triples where $0 \le a_i, b_i, c_i \le n$. We will subtract the $n^3$ triples where 0 doesn't appear and the $n^3$ triples where $n$ doesn't appear, and then add back the $(n-1)^3$ triples where 0 and $n$ don't appear, since they were subtracted twice. Thus,

$$f(n) = (n+1)^3 - 2n^3 + (n-1)^3 = 6n.$$

Now, we notice that $n = \nu_p(20!)$, or the exponent of the largest power of prime $p$ that divides $20!$, where $p$ is the prime that corresponds to $a_i$. Using Legendre's Formula, we find that

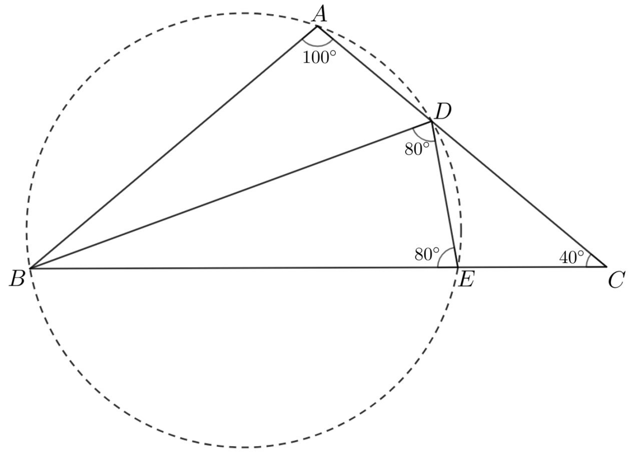
$$\nu_2(20!) = 18, \ \nu_3(20!) = 8, \ \nu_5(20!) = 4, \ \nu_7(20!) = 2,$$

and $\nu_p(20!) = 1$ if $p$ is a prime from 11 to 19 (inclusive). Thus,

$$\begin{aligned} N &= f(18) \cdot f(8) \cdot f(4) \cdot f(2) \cdot (f(1))^4 \\ &= 6^8 \cdot 18 \cdot 8 \cdot 4 \cdot 2 \cdot 1^4 \\ &= 2^{15} \cdot 3^{10}. \end{aligned}$$

So the number of factors of $N$ is $16 \times 11 = \boxed{176}$. □

**Problem 4.** Let $\triangle ABC$ be an isosceles triangle with $AB = AC$ and $\angle BAC = 100°$. Let point $D$ be on side $AC$ such that $BD$ bisects $\angle ABC$. Prove that $AD + DB = BC$.




*Solution 1.* Note that $m\angle ABC = m\angle ACB = 40°$, and $m\angle ABD = m\angle CBD = 20°$. Let point $E$ be on side $\overline{BC}$ such that $BD = BE$. Then $m\angle BDE = m\angle BED = 80°$, so $m\angle BAD + m\angle BED = 180°$.

We can now see that $ADEB$ is cyclic, so $\angle ABD \cong \angle EBD$ implies $ED = AD$. Also, $\triangle EDC \sim \triangle ABC$, so $EC = ED$. Then, since $EC + BE = BC$, it must be true that $AD + DB = BC$, and we are done. $\square$

*Solution 2.* WLOG let $BD = 1$. By the Law of Sines in $\triangle ABD$,

$$AD = \frac{\sin\angle ABD}{\sin\angle A} = \frac{\sin(20°)}{\sin(100°)} = \frac{\sin(20°)}{\sin(80°)}.$$

Similarly, by the Law of Sines in $\triangle CBD$,

$$BC = \frac{\sin\angle BDC}{\sin\angle C} = \frac{\sin(120°)}{\sin(40°)} = \frac{\sqrt{3}}{2\sin(40°)}.$$

By Sum-to-Product formulas,

$$\sin(20°) + \sin(80°) = 2\sin(50°)\cos(30°) = \sqrt{3}\sin(50°).$$

Now, we are ready to show that

$$\begin{aligned} AD + DB &= \frac{\sin(20°)}{\sin(80°)} + 1 \\ &= \frac{\sin(20°) + \sin(80°)}{\sin(2\cdot 40°)} \\ &= \frac{\sqrt{3}\sin(50°)}{2\sin(40°)\cos(40°)} \\ &= \frac{\sqrt{3}}{2\sin(40°)} = BC, \end{aligned}$$

as desired. $\square$

**Problem 5.** Let $n \in \mathbb{N}$ and $a_1 < a_2 < a_3 < \cdots < a_{\phi(n)}$ be the integers less than $n$ and relatively prime to $n$. Prove that $a_1 \cdot a_2 \cdot a_3 \cdots a_{\phi(n)} \equiv \pm 1 \pmod{n}$ for $n \geq 2$.
**Bonus.** For which $n$ is $a_1 \cdot a_2 \cdot a_3 \cdots a_{\phi(n)} \equiv -1 \pmod{n}$?

*Solution.* First, we must establish the existence of inverses modulo $n$.

**Lemma.** $\{a_1, a_2, a_3, \cdots, a_{\phi(n)}\} \equiv \{a_i a_1, a_i a_2, a_i a_3, \cdots, a_i a_{\phi(n)}\} \pmod{n}$ for all $1 \leq i \leq \phi(n)$.

*Proof.* $a_i a_j$ is coprime to $n$ since $a_i$ and $a_j$ are individually coprime to $n$. This implies that every element in $\{a_i a_1, a_i a_2, a_i a_3, \cdots, a_i a_{\phi(n)}\}$ corresponds with one of the elements in $\{a_1, a_2, a_3, \cdots, a_{\phi(n)}\}$ modulo $n$.

Also, $a_i a_j \not\equiv a_i a_k \pmod{n}$ for $1 \leq j, k \leq n$ and $j \neq k$. Assuming the contrary,

$$a_i a_j \equiv a_i a_k \pmod{n}$$

implies $a_i(a_j - a_k) \equiv 0 \pmod{n}$. Since $\gcd(a_i, n) = 1$, this means that $n \mid a_j - a_k$, so

$$a_j \equiv a_k \pmod{n}.$$

Thus, $j = k$, a contradiction.

Since the two sets are of the same size, and no two elements of

$$\{a_i a_1, a_i a_2, a_i a_3, \cdots, a_i a_{\phi(n)}\} \pmod{n}$$

are the same, the two sets are equivalent modulo $n$. This proves the lemma.

By the lemma, for every $a_i$, there exists an $a_j$ such that $a_i a_j \equiv 1 \pmod{n}$; we call $a_j$ the inverse of $a_i \pmod{n}$, also denoted as $a_i^{-1} \pmod{n}$.

Notice that if $a_j$ is the inverse of $a_i$, then $a_i$ is the inverse of $a_j$. Thus, inverses pair up and cancel out in the product $a_1 \cdot a_2 \cdot a_3 \cdots a_{\phi(n)}$. The only case left to handle is if $a_i$ is its own inverse, that is, the case where $a_i^2 \equiv 1 \pmod{n}$.

However, such $a_i$ also have their neat symmetry, as

$$a_i^2 \equiv 1 \pmod{n} \iff (-a_i)^2 \equiv 1 \pmod{n}.$$

Therefore, $a_i$ and $-a_i$ both belong to this set of self-inverses, and

$$a_i(-a_i) \equiv -a_i^2 \equiv -1 \pmod{n}.$$

It is importnt to note that $a_i \not\equiv -a_i \pmod{n}$ because if that were the case, then $a_i = \frac{n}{2}$, which is not relatively prime to $n$ for $n > 2$. For $n = 2$, it can be easily verified that $a_1 \equiv 1 \pmod{2}$, so the claim is true for $n = 2$. Hence, if we denote $k$ as the number of self-inverses modulo $n$, then

$$a_1 \cdot a_2 \cdot a_3 \cdots a_{\phi(n)} \equiv (-1)^{\frac{k}{2}} \equiv \pm 1 \pmod{n}$$

for $n > 2$, as desired. ∎

*Bonus Solution.* As previously shown,

$$a_1 \cdot a_2 \cdot a_3 \cdots a_{\phi(n)} \equiv (-1)^{\frac{k}{2}} \pmod{n},$$

so the problem statement is equivalent to finding $n$ such that $k \equiv 2 \pmod{4}$. (Recall that $k$ is the number of self-inverses modulo $n$.) Let

$$n = p_1^{k_1} p_2^{k_2} \cdots p_m^{k_m},$$

and let $S_i$ be the number of solutions to $x^2 \equiv 1 \pmod{i}$. By the Chinese Remainder Theorem, $k = S_n = S_{p_1^{k_1}} S_{p_2^{k_2}} \cdots S_{p_m^{k_m}}$.

**Case 1.** If $p_i$ is odd, then $S_{p_i^{k_i}} = 2$.
*Proof.* Given $x^2 \equiv 1 \pmod{p_i}$, we get that

$$x^2 - 1 \equiv (x-1)(x+1) \equiv 0 \pmod{p_i^{k_i}}.$$

Since $p_i$ is odd and therefore $p_i > 2$, either $p_i^{k_i} \mid x-1$ or $p_i^{k_i} \mid x+1$, since $p_i$ cannot divide both $x-1$ and $x+1$. That gives the two solutions

$$x \equiv 1, -1 \pmod{p_i^{k_i}},$$

so $S_{p_i^{k_i}} = 2$.

**Case 2.** If $p_i = 2$, then $S_2 = 1$, $S_4 = 2$, and $S_{2^{k_i}} = 4$ for $k_i > 2$.
*Proof.* $S_2$ and $S_4$ can be found manually as having solutions $x \equiv 1 \pmod{2}$ and $x \equiv 1, 3 \pmod{4}$, respectively. For $k_i > 2$, we find that

$$x^2 - 1 \equiv (x-1)(x+1) \equiv 0 \pmod{2^{k_i}}.$$

Now, $2^{k_i-1} \mid x-1$ or $2^{k_i-1} \mid x+1$ is sufficient since the other factor will contribute exactly one power of 2. This leads to the four solutions

$$x \equiv 1, 2^{k_i-1}-1, 2^{k_i-1}+1, -1 \pmod{2^{k_i}}.$$

Thus, $S_{2^{k_i}} = 4$.

From these two cases, we can find that

$$k = S_n = S_{p_1^{k_1}} S_{p_2^{k_2}} \cdots S_{p_m^{k_m}} \equiv 2 \pmod{4}$$

if and only if $n = 4$, $p^k$, or $2p^k$, as $S_n = 2$ in those scenarios. Furthermore, $n = 2$ is a trivial case with $S_2 = 1$. Hence,

$$a_1 \cdot a_2 \cdot a_3 \cdots a_{\phi(n)} \equiv -1 \pmod{n} \iff n = 2, 4, p^k, 2p^k,$$

as desired.

**Remark.** These solutions coincide with the set of integers $n$ for which there exists a primitive root $\pmod{n}$.

□