Team Solutions

1. Given a trapezoid with bases AB and CD, there exists a point E on CD such that drawing the segments AE and BE partitions the trapezoid into 3 similar isosceles triangles, each with long side twice the short side. What is the sum of all possible values of $\frac{CD}{AB}$?

   Proposed by Adam Bertelli

   Answer: $\dfrac{65}{4}$

   Solution: For each of the three triangles, we will call it vertical, or V, if the short side lies on one of the bases, and horizontal, or H, otherwise. From left to right, we can consider the value of $\frac{CD}{AB}$ over all possible sequences of triangle orientations:

   HHH/VVV: $\frac{1+1}{1} = 2$

   HHV/VHH: $\frac{4+1}{4} = \frac{5}{4}$

   HVV/VVH: $\frac{4+1}{1} = 5$

   HVH: $\frac{4+4}{1} = 8$

   VHV: impossible

   Thus in total, our possible values are $2 + \frac{5}{4} + 5 + 8 = \boxed{\dfrac{65}{4}}$.

2. Let $p_1, p_2, p_3, p_4, p_5, p_6$ be distinct primes greater than 5. Find the minimum possible value of

$$p_1 + p_2 + p_3 + p_4 + p_5 + p_6 - 6\min(p_1, p_2, p_3, p_4, p_5, p_6).$$

   Proposed by Oliver Hayman

   Answer: 46

   Solution: Assume $p_1 < p_2 < p_3 < p_4 < p_5 < p_6$. We have that $p_1 + p_2 + p_3 + p_4 + p_5 + p_6 - 6\min(p_1, p_2, p_3, p_4, p_5, p_6) = p_2 - p_1 + p_3 - p_1 + p_4 - p_1 + p_5 - p_1 + p_6 - p_1 = 5(p_2 - p_1) + 4(p_3 - p_2) + 3(p_4 - p_3) + 2(p_5 - p_4) + (p_6 - p_5)$.

   Note that $p_{i+1} - p_i$ is even for all i, as any prime greater than 5 is odd. Additionally, it is impossible for $p_{i+2} - p_{i+1} = p_{i+1} - p_i = 2$ for any i, as each of p_i, p_i+2, p_i+4 gives a different residue modulo 3, and no prime greater than 5 is divisible by 3. With only this criteria, clearly $(p_2 - p_1, p_3 - p_2, p_4 - p_3, p_5 - p_4, p_6 - p_5) = (2, 4, 2, 4, 2)$ minimizes the expression, which gives $5(p_2-p_1)+4(p_3-p_2)+3(p_4-p_3)+2(p_5-p_4)+(p_6-p_5) = 42$. However, this value cannot actually be achieved, as the numbers $p_1, p_1+2, p_1+6, p_1+8, p_1+12, p_1+14$ are congruent to $p_1, p_1+2, p_1+1, p_1+3, p_1+2, p_1+4$ mod 5, giving one of the numbers must be divisible by 5.

   Therefore, 42 is impossible to achieve, so the next value we consider is 44. However, the only possible sequence giving 44 would be $(p_2 - p_1, p_3 - p_2, p_4 - p_3, p_5 - p_4, p_6 - p_5) = (2, 4, 2, 4, 4)$, which is impossible to achieve as we have that $p_4, p_4 + 4, p_4 + 4 + 4$ give 3 different residues mod 3. Next we consider 46. This can be achieved if $(p_2 - p_1, p_3 - p_2, p_4 - p_3, p_5 - p_4, p_6 - p_5) = (2, 4, 2, 4, 6)$, which is possible if $(p_1, p_2, p_3, p_4, p_5, p_6) = (11, 13, 17, 19, 23, 29)$. Therefore, the smallest possible value is $\boxed{46}$.

3. Evaluate

$$\sum_{i=0}^{\infty} \frac{7^i}{(7^i+1)(7^i+7)}.$$

   Proposed by Connor Gordon

Answer: $\dfrac{7}{48}$

Solution: Let S be the desired sum. Divide numerator and denominator by 7 to get

$$S = \sum_{i=0}^{\infty} \frac{7^{i-1}}{(7^i+1)(7^{i-1}+1)}$$

The denominator of the summand looks promising for telescoping, so with some wishful thinking we set up

$$\frac{1}{7^{i-1}+1} - \frac{1}{7^i+1} = \frac{7^i - 7^{i-1}}{(7^i+1)(7^{i-1}+1)} = \frac{6(7^{i-1})}{(7^i+1)(7^{i-1}+1)}.$$

This means that

$$6S = \sum_{i=0}^{\infty} \left(\frac{1}{7^{i-1}+1} - \frac{1}{7^i+1} \right),$$

which clearly telescopes to $\frac{1}{7^{-1}+1} = \frac{7}{8}$. Diving by 6, we finally arrive at $S = \boxed{\frac{7}{48}}$.

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4. How many four-digit positive integers $\overline{a_1a_2a_3a_4}$ have only nonzero digits and have the property that $|a_i - a_j| \neq 1$ for all $1 \leq i < j \leq 4$?

Proposed by Kyle Lee

Answer: 2021

Solution: Note that the problem is equivalent to selecting four (not necessarily distinct) digits from $\{1,2,3,4,5,6,7,8,9\}$ satisfying the condition and then permuting them.

If there are 4 distinct digits, then a stars and bars argument shows that this is equivalent to finding the number of positive integer solutions to $a' + b + c + d + e' = 5 + 2$, which is $\binom{6}{4}$. So, there are a total of $\binom{6}{4} \cdot 4!$ solutions here.

If there are 3 distinct digits, then there are $\binom{7}{3} \cdot \frac{4!}{2} \cdot 3$ solutions via a similar argument.

If there are 2 distinct digits, then there are two subcases: numbers of the form $\overline{aabb}$ and $\overline{abbb}$. It can easily be seen that there are $\binom{8}{2} \cdot \frac{4!}{2!2!}$ and $\binom{8}{2} \cdot \frac{4!}{3!} \cdot 2$ solutions, respectively.

If there is 1 distinct digit, there are trivially 9 solutions.

The grand total is $\binom{6}{4} \cdot 4! + \binom{7}{3} \cdot \frac{4!}{2} \cdot 3 + \binom{8}{2} \cdot \frac{4!}{2!2!} + \binom{8}{2} \cdot \frac{4!}{3!} \cdot 2 + 9 = \boxed{2021}$.

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5. Let N be the fifth largest number that can be created by combining 2021 1's using addition, multiplication, and exponentiation, in any order (parentheses are allowed). If $f(x) = \log_2(x)$, and k is the least positive integer such that $f^k(N)$ is not a power of 2, what is the value of $f^k(N)$?

(Note: $f^k(N) = f(f(\cdots(f(N))))$, where f is applied k times.)

Proposed by Adam Bertelli

Answer: 3^{27}

Solution: First, let's define $M(x)$ to be the largest number we can make out of only 1's. The first few values of $M(n)$, starting from $n = 1$, are $1, 2, 3, 4, 9(\text{or } 3^2), 27(\text{or } 3^3)$, and it is clear that after this point we will have $M(x+2) = 2^{M(x)}$, since this is the most efficient way to increase our number for large enough $M(x)$.

Now, this tells us that $M(2021)$ will be a tower of $2^{2^{2^{\cdots}}}$ with 1007 2's and then $M(7)$ at the very top. On the other hand, the largest number that we can make without first having a tower of 1007 2's, would be a tower of 1006 2's, followed by $3^{M(6)}$ at the very top (since the further down we change something, the smaller our number will end up being, because for large numbers, having a larger exponent is way more important than a larger base). Let

$$B = 2^{\cdot^{\cdot^{\cdot^{3^{M(6)}}}}}$$

be the number we just described. Then, our solution will first consider how many numbers we can form using 1007 2's and some expression of 7 1's at the top, and only when these become smaller than B do we have to begin considering other options.

We can pretty easily list out the largest few possibilities for our expression using 7 1's as

$$2^{3^2}, 2^{2^3}, 3^4, 4^3 (\text{or } 2^{2\cdot 3}), 2^5, \dots \quad \text{or in other words} \quad 512, 256, 81, 64, 32, \dots$$

Now, we would like to compare these values (stacked on top of 1007 2's) to B. We can do this by noting that $3^{M(6)}$, or 3^{27}, satisfies

$$2^{32} < 8^{11} < 9^{13} < 3^{27}$$

and

$$3^{27} < 4^{32} = 2^{64}$$

so B lies between the 4th and 5th elements on our list. This tells us that the 5th largest possible number we can create is in fact B, so our answer is simply the result of repeatedly taking $\log_2$ of B until it is no longer a power of 2. Each application of $\log_2$ corresponds to removing a 2 from our stack of 2's, so once we have removed all of them, the remaining number is simply $3^{f(6)}$, or $\boxed{3^{27}}$.

Note: If B happened to come before the 4th number, and we ended up having to check other options, we could simply repeat this argument with $2^{f(9)}$ and $3^{f(8)}$, and so on until we could make sure we didn't need to consider any other values. However, the number of possible expressions near the top grows exponentially, so each step would take longer and longer to check.

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6. Let $P(x), Q(x)$, and $R(x)$ be three monic quadratic polynomials with only real roots, satisfying

$$P(Q(x)) = (x-1)(x-3)(x-5)(x-7)$$

$$Q(R(x)) = (x-2)(x-4)(x-6)(x-8)$$

for all real numbers x. What is $P(0) + Q(0) + R(0)$?

Proposed by Kyle Lee

Answer: 129

Solution: Note that since $P(Q(x))$ has distinct four roots, $Q(1), Q(3), Q(5), Q(7)$ must map to at most exactly two distinct values. It follows that $Q(x)$ is symmetric about $x = 4$, so $Q(x) = (x-4)^2 + a$, for some constant a. Plugging $x = 1, 3, 5, 7$ into $Q(x)$ gives the two distinct values of $a + 1$ and $a + 9$. Since $P(a + 1) = P(a + 9) = 0$, it follows that $P(x)$ is symmetric about $x = a + 5$, so $P(x) = (x - (a + 5))^2 + b$, for some constant b.

Doing the same thing on $Q(R(x))$, we have that $R(x) = (x-5)^2 + c$ and $Q(x) = (x - (c + 5))^2 + d$, for some constants c and d. But, we found earlier that $Q(x) = (x-4)^2 + a$, so $2(c + 5) = 8$ and $c = -1$. Therefore, $R(x) = (x-4)(x-6)$. Plugging in $x = 2, 4, 6, 8$ into $R(x)$ gives the two distinct values of 0 and 8, so $Q(x) = x(x-8)$. Lastly, plugging in $x = 1, 3, 5, 7$ into $Q(x)$ gives the two distinct values of -7 and -15, so $P(x) = (x+7)(x+15)$. In summary, we have found that $P(x) = x^2 + 22x + 105, Q(x) = x^2 - 8x$, and $x^2 - 10x + 24$, so the requested sum is $105 + 0 + 24 = \boxed{129}$.

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7. Let P and Q be fixed points in the Euclidean plane. Consider another point O_0. Define O_{i+1} as the center of the unique circle passing through O_i, P and Q. (Assume that O_i, P, Q are never collinear.) How many possible positions of O_0 satisfy that $O_{2021} = O_0$?

Proposed by Fei Peng

Answer: $2^{2021} - 2$

Solution: Say $P = (0, 1)$ and $Q = (0, -1)$. Then every $O_{\geq 1}$ is on the x-axis, so say $O_0 = (x, 0)$. Observe that if $O_i = (\cot(\alpha), 0)$, then $O_{i+1} = (\cot(2\alpha), 0)$. Thus, we are essentially finding the number of $\alpha \in (0, \pi)$ such that $x = \cot(\alpha) = \cot(2^{2021}\alpha)$. That is, $k := (2^{2021} - 1)\alpha/\pi$ should be an integer. In this range of α, $0 < k < 2^{2021} - 1$, so there are $\boxed{2^{2021} - 2}$ choices of k, each corresponding to a position of O_0.

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8. Determine the number of functions f from the integers to $\{1, 2, \cdots, 15\}$ which satisfy

$$f(x) = f(x + 15)$$

and

$$f(x + f(y)) = f(x - f(y))$$

for all x, y.

Proposed by Vijay Srinivasan

Answer: 375

Solution: Let $n = \gcd(f(0), f(1), f(2), \cdots, f(14), 15)$, so that $n \in \{1, 3, 5, 15\}$. Since n can be written as a sum of multiples of numbers of the form $f(y)$, it follows that $f(x) = f(x + n)$ for all x. We case on the value of n.

If $n = 1$, then f is a constant function with some value c, and $\gcd(c, 15) = 1$. There are 8 such values of c in the set $\{1, 2, \ldots, 15\}$.

If $n = 3$, then f is determined by $f(0), f(1), f(2)$, subject to the constraint $\gcd(f(0), f(1), f(2), 15) = 3$. We see that $(f(0), f(1), f(2))$ must consist of multiples of 3, and they cannot all be equal to 15. There are thus $5^3 - 1$ possible f from this case.

If $n = 5$, proceed as in the previous case to conclude that there are $3^5 - 1$ possible f.

If $n = 15$, then clearly $f(0) = f(1) = \cdots = f(14) = 15$, so there is only one possible function f.

The final count is $8 + (3^5 - 1) + (5^3 - 1) + 1 = \boxed{375}$.

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9. Let ABC be a triangle with circumcenter O. Additionally, $\angle BAC = 20^\circ$ and $\angle BCA = 70^\circ$. Let D, E be points on side AC such that BO bisects $\angle ABD$ and BE bisects $\angle CBD$. If P and Q are points on line BC such that DP and EQ are perpendicular to AC, what is $\angle PAQ$?

Proposed by Daniel Li

Answer: 25

Solution: Note that ABC is a right triangle. Since O is its circumcenter, triangle AOB is isosceles with $\angle ABO = \angle BAO = 20^\circ$. This means $\angle DBO = 25^\circ$, so $\angle EBD = \angle EBC = 25^\circ$.

Since $\angle ABP = 90^\circ = \angle ADP$, $ADBP$ is cyclic. Similarly, $AEBQ$ is cyclic. Therefore, we have

$$\begin{aligned}
\angle PAQ &= \angle PAD - \angle QAE \\
&= \angle CBD - \angle CBE \\
&= \angle EBD \\
&= \boxed{25^\circ}
\end{aligned}$$

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10. How many functions $f : \{1, 2, 3, \ldots, 7\} \to \{1, 2, 3, \ldots, 7\}$ are there such that the set $\mathcal{F} = \{f(i) : i \in \{1, \ldots, 7\}\}$ has cardinality four, while the set $\mathcal{G} = \{f(f(f(i))) : i \in \{1, \ldots, 7\}\}$ consists of a single element?

Proposed by Sam Delatore

Answer: 23520

Solution: Let $\mathcal{H} = \{f(f(i)) : i \in \{1, \ldots, 7\}\}$. Clearly $\mathcal{F} \supseteq \mathcal{H} \supseteq \mathcal{G}$.

Case 1: $|\mathcal{H}| = 4$.

Suppose without loss of generality that $\mathcal{F} = \mathcal{H} = \{1, 2, 3, 4\}$. Then, f permutes the values from 1 to 4, giving $\mathcal{G} = \{1, 2, 3, 4\}$, a contradiction.

Case 2: $|\mathcal{H}| = 3$.

Suppose $\mathcal{F} = \{1, 2, 3, 4\}$ and $\mathcal{H} = \{1, 2, 3\}$. Further suppose $f(4) = 1$. Since $2, 3 \in \mathcal{H}$, we must have $f(1), f(2)$ and $f(3)$ cover 2 and 3. This places $2, 3 \in \mathcal{G}$, contradiction.

Case 3: $|\mathcal{H}| = 2$.

Suppose $\mathcal{F} = \{1, 2, 3, 4\}$, $\mathcal{H} = \{1, 2\}$, $\mathcal{G} = \{1\}$. Then, $f(1) = f(2) = 1$ is forced. At least one of $f(3)$ or $f(4)$ must be 2, with three options. Finally, we need $f(5), f(6)$, and $f(7)$ to cover 3 and 4; 18 choices satisfy this. There are $\binom{7}{4}\binom{4}{2}\binom{2}{1} = 420$ ways to choose $\mathcal{F}$, $\mathcal{H}$, and $\mathcal{G}$, each giving $3 \cdot 18 = 54$ different functions. There are $54 \cdot 420 = 22680$ total functions in this case.

Case 4: $|\mathcal{H}| = 1$.

Suppose $\mathcal{F} = \{1, 2, 3, 4\}$, $\mathcal{H} = \mathcal{G} = \{1\}$. $f(1) = f(2) = f(3) = f(4) = 1$ is forced. We must have $f(5), f(6)$, and $f(7)$ cover 2, 3, and 4; six choices satisfy this.

There are $\binom{7}{4}\binom{4}{1}\binom{1}{1} = 140$ ways to choose $\mathcal{F}$, $\mathcal{H}$, and $\mathcal{G}$, each giving six different functions. There are $6 \cdot 140 = 840$ total functions in this case.

The answer is $0 + 0 + 22680 + 840 = \boxed{23520}$

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11. The set of all points (x, y) in the plane satisfying $x < y$ and $x^3 - y^3 > x^2 - y^2$ has area A. What is the value of A?

Proposed by Adam Bertelli

Answer: $\dfrac{\pi\sqrt{3}}{9}$

Solution: We are given that $x - y < 0$, so dividing through the inequality $x^3 - y^3 > x^2 - y^2$ by $x - y$ and rearranging gives

$$x^2 + xy + y^2 < x + y \implies f(x, y) < 0$$

where $f(x, y) = x^2 - x + xy - y + y^2$. Clearly f is an irreducible quadratic, and for $|x|, |y|$ large, we have $f > 0$, so the inequality $f < 0$ defines the interior of an ellipse E. Since f is symmetric in x and y, the line $y = x$ is an axis of E. Plugging in $y = x$ to $f(x, y) = 0$, we obtain $3x^2 - 2x = 0$, so the intersections of this axis with E occur at $x = 0$ and $x = \frac{2}{3}$. This implies that E is centered at $\left(\frac{1}{3}, \frac{1}{3}\right)$ and has one of its radii equal to $\frac{\sqrt{2}}{3}$.

Since the other axis is perpendicular to the first and passes through the center, it is given by $y = \frac{2}{3} - x$. Observe that $f(x, y) = (x + y)^2 - (x + y) - xy$, so plugging in $x + y = \frac{2}{3}$ to $f(x, y) = 0$ gives

$$0 = f\left(x, \frac{2}{3} - x\right) = -\frac{2}{9} - x\left(\frac{2}{3} - x\right) = x'^2 - \frac{1}{3}$$

where $x' = \frac{1}{3} - x$ measures both the horizontal and vertical distance to the center. We find that the solutions are $x' = \pm\frac{\sqrt{3}}{3}$, so the other radius of E is $\frac{\sqrt{3} \cdot \sqrt{2}}{3}$.

Since we only want the area of the half of the ellipse where $x < y$, we compute a final answer of

$$\frac{1}{2} \cdot \pi \cdot \frac{\sqrt{2}}{3} \cdot \frac{\sqrt{3} \cdot \sqrt{2}}{3} = \boxed{\frac{\pi\sqrt{3}}{9}}.$$

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12. Let $\triangle ABC$ be a triangle, and let l be the line passing through its incenter and centroid. Assume that B and C lie on the same side of l, and that the distance from B to l is twice the distance from C to l. Suppose also that the length BA is twice that of CA. If $\triangle ABC$ has integer side lengths and is as small as possible, what is $AB^2 + BC^2 + CA^2$?

Proposed by Thomas Lam

Answer: 61

Solution: This solution uses the following facts from physics: A set of point-masses on a plane is balanced at the centroid, and furthermore is balanced along any axis that passes through the centroid. Also, if it is balanced along an axis, then the torques induced by the masses must cancel.

Now let the distances from A, B, C to l be x, y, z, respectively, and let BC, CA, AB be a, b, c, respectively. Assign A, B, C equal masses so that the centroid of the system is G. Since l passes through G, the system is balanced about the axis l, and hence the torques induced by A, B, and C must sum to zero. Hence, since we are given that B and C lie on the same side of l, we may write $x = y + z$.

Next, assign masses of a, b, c to A, B, C, so that the system is balanced over the incenter. Again, l passes through I, so the system is balanced about the axis l. Now we may write $ax = by + cz$.

Combining the equations and eliminating x, we obtain $y(b - a) = z(a - c)$. But we are given that $y = 2z$, so $2(b - a) = (a - c)$ and $3a = 2b + c$. We are also given that $c = 2b$, thus $3a = 4b = 2c$. The smallest solution to this equation in positive integers is given by $(a, b, c) = (4, 3, 6)$, thus the desired answer is $4^2 + 3^2 + 6^2 = \boxed{61}$.

Alternate Solution:

We use Barycentric coordinates. Consider B', the reflection of B over C. Then B, I, O are collinear by the distance condition. We have $B' = (0 : 1 : -2)$, $I = (a, b, c)$, and $G = (1 : 1 : 1)$. Thus we require:

$$\begin{vmatrix} 0 & 1 & -2 \\ a & b & c \\ 1 & 1 & 1 \end{vmatrix} = 0$$

This reduces to $3a = 2b + c$, and we may proceed as before.

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13. Let $p = 3 \cdot 10^{10} + 1$ be a prime and let p_n denote the probability that $p \mid (k^k - 1)$ for a random k chosen uniformly from $\{1, 2, \cdots, n\}$. Given that $p_n \cdot p$ converges to a value L as n goes to infinity, what is L?

Proposed by Vijay Srinivasan

Answer: 90

Solution: First observe that $a^m \bmod p$ is uniquely determined by the values of $a \bmod p$ and $m \bmod (p-1)$. It follows that $k^k \bmod p$ is uniquely determined by the value of $k \bmod p(p-1)$. It follows that $L = p \cdot p_n$ for $n = p(p-1)$.

Let a be an element of $\mathbb{Z}/p\mathbb{Z}$, and let d be the order of a modulo p. If k is a positive integer with $k \equiv a \bmod p$, then $k^k \equiv 1 \bmod p$ if and only if $k \equiv 0 \bmod d$. It follows that there are exactly $\frac{p-1}{d}$ residue classes modulo $p(p-1)$ for which $k^k \equiv 1 \bmod p$ and $k \equiv a \bmod p$. For a given residue class $a \bmod p$, there is then a $1/d$

chance of having $k \equiv a \bmod p$ satisfy $k^k \equiv 1 \bmod p$, with k chosen uniformly modulo $p-1$. Since, for each $d \mid (p-1)$, there are $\phi(d)$ elements of $\mathbb{Z}/p\mathbb{Z}$ of order d, we get

$$L = p \cdot \frac{1}{p(p-1)} \sum_{d \mid p-1} \phi(d) \cdot \frac{p-1}{d} = \sum_{d \mid p-1} \frac{\phi(d)}{d}.$$

For ease we let $f(b) = \sum_{d \mid b} \frac{\phi(d)}{d}$, which is a multiplicative function. To compute $L = f(p-1)$ it suffices to compute f on prime powers. For a prime power q^m, with $m > 0$, we have $\phi(q^m)/q^m = 1 - 1/q$; it follows that

$$f(q^m) = 1 + m - \frac{m}{q}.$$

Finally, for $p - 1 = 3 \cdot 2^{10} \cdot 5^{10}$, we get

$$f(p-1) = \left(1 + 1 - \frac{1}{3}\right)\left(1 + 10 - \frac{10}{2}\right)\left(1 + 10 - \frac{10}{5}\right) = \boxed{90.}$$

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14. Let S be the set of lattice points $(x, y) \in \mathbb{Z}^2$ such that $-10 \le x, y \le 10$. Let the point $(0,0)$ be O. Let Scotty the Dog's position be point P, where initially $P = (0,1)$. At every second, consider all pairs of points $C, D \in S$ such that neither C nor D lies on line OP, and the area of quadrilateral $OCPD$ (with the points going clockwise in that order) is 1. Scotty finds the pair C, D maximizing the sum of the y coordinates of C and D, and randomly jumps to one of them, setting that as the new point P. After 50 such moves, Scotty ends up at point $(1,1)$. Find the probability that he never returned to the point $(0,1)$ during these 50 moves.

Proposed by David Tang

Answer: $\frac{16}{25}$

Solution: We notice that areas of triangle OCP and ODP are both multiples of $1/2$ by shoelace on lattice points, so both have to be $1/2$, since none can be 0 by the collinearity condition. Let $C = (x_c, y_c)$, $D = (x_d, y_d)$, $P = (x_p, y_p)$. By shoelace, we have that C, D are points such that $y_c x_p - y_p x_c = 1$ and $y_d x_p - y_p x_d = -1$. We find that these are equivalent to moving up and down terms in the degree 10 Farey sequence, restricted to the $y > 0$ half plane (at least for the first 50 moves). Another way to see this is that, given line OP, the points we can move to are the first points we hit when sweeping this line clockwise or counterclockwise, centered around O.

Now, we want to find the position of the term $(1,1)$ in this sequence. This clearly occurs after every point in the upper half triangle of the first quadrant, so we count the number of such points as $\phi(1)+\phi(2)+\cdots+\phi(10) = 31$, meaning $(1,1)$ is the 32nd point in our sequence. This tells us that in total, Scotty went up in our sequence 41 times, and down 9 times, so by the Ballot Theorem, we see that the probability that we never reach the starting point again is $\frac{41-9}{41+9} = \boxed{\frac{16}{25}}$.

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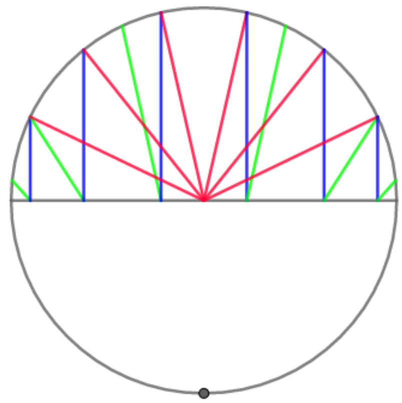
15. Adam has a circle of radius 1 centered at the origin.

- First, he draws 6 segments from the origin to the boundary of the circle, which splits the upper (positive y) semicircle into 7 equal pieces.

- Next, starting from each point where a segment hit the circle, he draws an altitude to the x-axis.

- Finally, starting from each point where an altitude hit the x-axis, he draws a segment directly away from the bottommost point of the circle $(0,-1)$, stopping when he reaches the boundary of the circle.

What is the product of the lengths of all 18 segments Adam drew?



Proposed by Adam Bertelli

Answer: $\frac{7^3}{2^{12}13^2}$

Solution: The 6 radii have unit length and so can be ignored. For $k = 1, 2, \ldots, 6$, let $\theta_k = \frac{\pi k}{7}$. Let b_k and g_k denote the lengths of the kth blue and green segments from the right, respectively. Clearly $b_k = \sin\theta_k$. The distance from $(0, -1)$ to the foot of the kth blue segment is $\sqrt{1 + \cos^2\theta_k}$ by the Pythagorean theorem, so Power of a Point says that $g_k\sqrt{1 + \cos^2\theta_k} = b_k^2$. It follows that the desired product P is given by

$$P = \prod_{k=1}^{6} b_k g_k = \prod_{k=1}^{6} \frac{\sin^3\theta_k}{\sqrt{1 + \cos^2\theta_k}}.$$

Let $Q = \prod_{k=1}^{6} \sin\theta_k$ and $R = \prod_{k=1}^{6} \sqrt{1 + \cos^2\theta_k}$.
To evaluate Q, we write

$$Q = \prod_{k=1}^{3} \sin^2\theta_k = \prod_{k=1}^{3} \frac{1 - \cos 2\theta_k}{2} = \frac{1}{2^6}\prod_{k=1}^{3}(2 - 2\cos 2\theta_k).$$

Let $A(x)$ be the monic cubic polynomial with roots $2\cos 2\theta_k$ for $k = 1, 2, 3$, so that $Q = \frac{A(2)}{2^6}$.
To evaluate R, we write $\cos^2\theta_k = \frac{1+\cos 2\theta_k}{2}$ so we have

$$R = \prod_{k=1}^{6} \sqrt{1 + \cos^2\theta_k} = \prod_{k=1}^{3}(1 + \cos^2\theta_k) = \prod_{k=1}^{3} \frac{3 + \cos 2\theta_k}{2} = \frac{1}{2^6}\prod_{k=1}^{3}(6 + 2\cos 2\theta_k)$$

so $R = -\frac{A(-6)}{2^6}$. We conclude that

$$P = \frac{Q^3}{R} = -\frac{A(2)^3}{2^{12}A(-6)}$$

It remains to compute the polynomial A. Observe that A is the minimal polynomial of $\zeta + \zeta^{-1}$, where $\zeta = e^{2i\theta_1}$. The minimal polynomial of ζ is $x^6 + x^5 + \cdots + 1$. Substituting $y = x + x^{-1}$, this becomes $x^3(y^3 + y^2 - 2y - 1)$, so $A(y) = y^3 + y^2 - 2y - 1$. We now evaluate $A(2) = 7$ and $A(-6) = -169$ and so obtain the desired answer.