CMIMC 2021

Team

1. Given a trapezoid with bases AB and CD, there exists a point E on CD such that drawing the segments AE and BE partitions the trapezoid into 3 similar isosceles triangles, each with long side twice the short side. What is the sum of all possible values of $\frac{CD}{AB}$?

2. Let $p_1, p_2, p_3, p_4, p_5, p_6$ be distinct primes greater than 5. Find the minimum possible value of
$$p_1 + p_2 + p_3 + p_4 + p_5 + p_6 - 6\min(p_1, p_2, p_3, p_4, p_5, p_6).$$

3. Evaluate
$$\sum_{i=0}^{\infty} \frac{7^i}{(7^i+1)(7^i+7)}.$$

4. How many four-digit positive integers $\overline{a_1a_2a_3a_4}$ have only nonzero digits and have the property that $|a_i - a_j| \neq 1$ for all $1 \leq i < j \leq 4$?

5. Let N be the fifth largest number that can be created by combining 2021 1's using addition, multiplication, and exponentiation, in any order (parentheses are allowed). If $f(x) = \log_2(x)$, and k is the least positive integer such that $f^k(N)$ is not a power of 2, what is the value of $f^k(N)$? (Note: $f^k(N) = f(f(\cdots(f(N))))$, where f is applied k times.)

6. Let $P(x), Q(x)$, and $R(x)$ be three monic quadratic polynomials with only real roots, satisfying
$$P(Q(x)) = (x-1)(x-3)(x-5)(x-7)$$
$$Q(R(x)) = (x-2)(x-4)(x-6)(x-8)$$
for all real numbers x. What is $P(0) + Q(0) + R(0)$?

7. Let P and Q be fixed points in the Euclidean plane. Consider another point O_0. Define O_{i+1} as the center of the unique circle passing through O_i, P and Q. (Assume that O_i, P, Q are never collinear.) How many possible positions of O_0 satisfy that $O_{2021} = O_0$?

8. Determine the number of functions f from the integers to $\{1, 2, \cdots, 15\}$ which satisfy
$$f(x) = f(x+15)$$
and
$$f(x + f(y)) = f(x - f(y))$$
for all x, y.

9. Let ABC be a triangle with circumcenter O. Additionally, $\angle BAC = 20^\circ$ and $\angle BCA = 70^\circ$. Let D, E be points on side AC such that BO bisects $\angle ABD$ and BE bisects $\angle CBD$. If P and Q are points on line BC such that DP and EQ are perpendicular to AC, what is $\angle PAQ$?

10. How many functions $f : \{1, 2, 3, \ldots, 7\} \to \{1, 2, 3, \ldots, 7\}$ are there such that the set $\mathcal{F} = \{f(i) : i \in \{1, \ldots, 7\}\}$ has cardinality four, while the set $\mathcal{G} = \{f(f(f(i))) : i \in \{1, \ldots, 7\}\}$ consists of a single element?

11. The set of all points (x, y) in the plane satisfying $x < y$ and $x^3 - y^3 > x^2 - y^2$ has area A. What is the value of A?

12. Let $\triangle ABC$ be a triangle, and let l be the line passing through its incenter and centroid. Assume that B and C lie on the same side of l, and that the distance from B to l is twice the distance from C to l. Suppose also that the length BA is twice that of CA. If $\triangle ABC$ has integer side lengths and is as small as possible, what is $AB^2 + BC^2 + CA^2$?

13. Let $p = 3 \cdot 10^{10} + 1$ be a prime and let p_n denote the probability that $p \mid (k^k - 1)$ for a random k chosen uniformly from $\{1, 2, \cdots, n\}$. Given that $p_n \cdot p$ converges to a value L as n goes to infinity, what is L?

14. Let S be the set of lattice points $(x, y) \in \mathbb{Z}^2$ such that $-10 \leq x, y \leq 10$. Let the point $(0, 0)$ be O. Let Scotty the Dog's position be point P, where initially $P = (0, 1)$. At every second, consider all pairs of points $C, D \in S$ such that neither C nor D lies on line OP, and the area of quadrilateral $OCPD$ (with the points going clockwise in that order) is 1. Scotty finds the pair C, D maximizing the sum of the y coordinates of C and D, and randomly jumps to one of them, setting that as the new point P. After 50 such moves, Scotty ends up at point $(1, 1)$. Find the probability that he never returned to the point $(0, 1)$ during these 50 moves.

15. Adam has a circle of radius 1 centered at the origin.

    - First, he draws 6 segments from the origin to the boundary of the circle, which splits the upper (positive y) semicircle into 7 equal pieces.

    - Next, starting from each point where a segment hit the circle, he draws an altitude to the x-axis.

    - Finally, starting from each point where an altitude hit the x-axis, he draws a segment directly away from the bottommost point of the circle $(0, -1)$, stopping when he reaches the boundary of the circle.

    What is the product of the lengths of all 18 segments Adam drew?