

HMMT November 2021

November 13, 2021

Team Round

- [20] Let $ABCD$ be a parallelogram. Let E be the midpoint of AB and F be the midpoint of CD . Points P and Q are on segments EF and CF , respectively, such that A , P , and Q are collinear. Given that $EP = 5$, $PF = 3$, and $QF = 12$, find CQ .
- [25] Joey wrote a system of equations on a blackboard, where each of the equations was of the form $a + b = c$ or $a \cdot b = c$ for some variables or integers a, b, c . Then Sean came to the board and erased all of the plus signs and multiplication signs, so that the board reads:

$$\begin{aligned}x & z = 15 \\x & y = 12 \\x & x = 36\end{aligned}$$

If x, y, z are integer solutions to the original system, find the sum of all possible values of $100x + 10y + z$.

- [30] Suppose m and n are positive integers for which
 - the sum of the first m multiples of n is 120, and
 - the sum of the first m^3 multiples of n^3 is 4032000.

Determine the sum of the first m^2 multiples of n^2 .

- [35] Find the number of 10-digit numbers $\overline{a_1a_2\cdots a_{10}}$ which are multiples of 11 such that the digits are non-increasing from left to right, i.e. $a_i \geq a_{i+1}$ for each $1 \leq i \leq 9$.
- [40] How many ways are there to place 31 knights in the cells of an 8×8 unit grid so that no two attack one another?
(A knight attacks another knight if the distance between the centers of their cells is exactly $\sqrt{5}$.)
- [40] The *taxicab distance* between points (x_1, y_1) and (x_2, y_2) is $|x_2 - x_1| + |y_2 - y_1|$. A regular octagon is positioned in the xy plane so that one of its sides has endpoints $(0, 0)$ and $(1, 0)$. Let S be the set of all points inside the octagon whose taxicab distance from some octagon vertex is at most $\frac{2}{3}$. The area of S can be written as $\frac{m}{n}$, where m, n are positive integers and $\gcd(m, n) = 1$. Find $100m + n$.
- [45] Let $f(x) = x^3 + 3x - 1$ have roots a, b, c . Given that

$$\frac{1}{a^3 + b^3} + \frac{1}{b^3 + c^3} + \frac{1}{c^3 + a^3}$$

can be written as $\frac{m}{n}$, where m, n are positive integers and $\gcd(m, n) = 1$, find $100m + n$.

- [50] Paul and Sara are playing a game with integers on a whiteboard, with Paul going first. When it is Paul's turn, he can pick any two integers on the board and replace them with their product; when it is Sara's turn, she can pick any two integers on the board and replace them with their sum. Play continues until exactly one integer remains on the board. Paul wins if that integer is odd, and Sara wins if it is even.

Initially, there are 2021 integers on the board, each one sampled uniformly at random from the set $\{0, 1, 2, 3, \dots, 2021\}$. Assuming both players play optimally, the probability that Paul wins is $\frac{m}{n}$, where m, n are positive integers and $\gcd(m, n) = 1$. Find the remainder when $m + n$ is divided by 1000.

- [55] Let N be the smallest positive integer for which

$$x^2 + x + 1 \text{ divides } 166 - \sum_{d|N, d>0} x^d.$$

Find the remainder when N is divided by 1000.

10. [60] Three faces \mathcal{X} , \mathcal{Y} , \mathcal{Z} of a unit cube share a common vertex. Suppose the projections of \mathcal{X} , \mathcal{Y} , \mathcal{Z} onto a fixed plane \mathcal{P} have areas x , y , z , respectively. If $x : y : z = 6 : 10 : 15$, then $x + y + z$ can be written as $\frac{m}{n}$, where m, n are positive integers and $\gcd(m, n) = 1$. Find $100m + n$.