Stuyvesant Team Contest: Solutions

Spring 2021

Problem 1. [**6**] The number $\overline{2111x}$ is divisible by exactly one 1-digit number d, where x is a digit. What is d?

Answer. $\boxed{1}$ *Proposed by Rishabh Das*

Solution. Every positive integer is divisible by 1, so if a number is divisible by exactly one single digit number, it must be 1. □

Problem 2. [**6**] If $a + b = 2021$ and $ab + b^2 = 20210$, compute $a + b^2$.

Answer. $\boxed{2111}$ *Proposed by Rishabh Das*

Solution. The second equation can be rewritten as

$$b(a + b) = 20210 \implies 2021b = 20210,$$

so $b = 10$. This means $a = 2021 - 10 = 2011$, so $a + b^2 = 2111$. □

Problem 3. [**7**] An isosceles triangle ABC has one angle equal to two times another angle. What is the sum of all possible values of $\angle A$, in degrees?

Answer. $\boxed{243}$ *Proposed by Rishabh Das*

Solution. The angles of the triangle are either $2x, 2x, x$ or $x, x, 2x$. In the first case, $5x = 180°$, so $x = 36°$. Then $\angle A$ can either be $36°$ or $72°$. In the second case, $4x = 180°$, so $x = 45°$. Then $\angle A$ can either be $45°$ or $90°$. Overall, the sum of the four cases is $36° + 72° + 45° + 90° = 243°$. □

Problem 4. [**7**] How many distinct four-digit numbers are there whose digits are a permutation of the digits of 2021 (including 2021 itself)?

Answer. $\boxed{9}$ *Proposed by Rishabh Das*

Solution. There are $\frac{24}{2} = 12$ permutations of the digits $2, 0, 2, 1$. In $\frac{1}{4}$ of these, 0 will be the first digit, thus making the number not have four digits. Subtracting these off, there are $12 - 3 = 9$ such numbers. □

Problem 5. [**8**] If x, y, and z are real numbers such that

$$(x + y)z = 16,$$
$$(y + z)x = 21, \text{ and}$$
$$(z + x)y = 25,$$

then compute $x^2 + y^2 + z^2$.

Answer. $\boxed{38}$ *Proposed by Rishabh Das*

Solution. Adding all of the equations yields

$$2(xy + yz + zx) = 62 \implies xy + yz + zx = 31.$$

Now subtracting each of the original equations from this gives $xy = 15, yz = 10$, and $zx = 6$. From here, multiplying these three equations gives

$$x^2y^2z^2 = 15 \cdot 10 \cdot 6 = 30^2 \implies xyz = \pm 30.$$

Dividing this by our three new equations gives $z = \pm 2, x = \pm 3$, and $y = \pm 5$. The answer is $2^2 + 3^2 + 5^2 = 38$. □

Problem 6. **[8]** If a, b, c, d, and e are positive integers satisfying

$$abcd : abce : abde : acde : bcde = 1 : 2 : 3 : 4 : 5$$

then find the smallest possible value of $a + b + c + d + e$.

Answer. $\boxed{137}$ *Proposed by Rishabh Das*

Solution. After dividing each of the terms on the left by $abcde$, the result is

$$\frac{1}{e} : \frac{1}{d} : \frac{1}{c} : \frac{1}{b} : \frac{1}{a} = 1 : 2 : 3 : 4 : 5.$$

Reciprocating all terms,

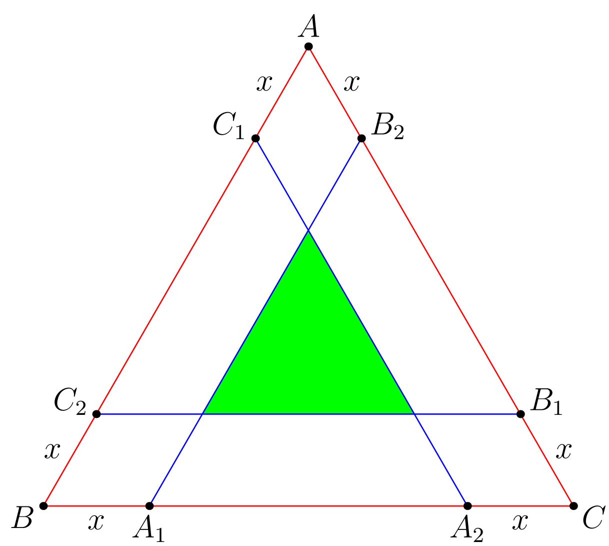
$$e : d : c : b : a = 1 : \frac{1}{2} : \frac{1}{3} : \frac{1}{4} : \frac{1}{5}.$$

Multiplying each of the terms on the right by 60, we see

$$e : d : c : b : a = 60 : 30 : 20 : 15 : 12.$$

The smallest possible value of $a + b + c + d + e$ is $60 + 30 + 20 + 15 + 12 = 137$. □

Problem 7. **[9]** Equilateral triangle ABC has side length 1. Points A_1 and A_2 are selected on side BC such that $BA_1 = CA_2 = x$, where $x < \frac{1}{2}$. Points B_1, B_2, C_1, and C_2 are defined similarly. Lines A_1B_2, B_1C_2, and C_1A_2 define another triangle, with area $\frac{1}{4}$ of the area of ABC. Compute x.



Answer. $\boxed{\frac{1}{6}}$ *Proposed by Rishabh Das*

Solution. The side length of the green triangle will be $\sqrt{\frac{1}{4}} = \frac{1}{2}$ the side length of $\triangle ABC$, or just $\frac{1}{2}$. $\triangle BC_1A_2$ is equilateral, so $BA_2 = A_2C_1$. This means

$$1 - x = \frac{1}{2} + 2x,$$

so $x = \dfrac{1}{6}$. □

Problem 8. **[9]** There are n consecutive positive integers, each of which can be written as the sum of two primes. What is the maximum possible value of n?

Answer. $\boxed{7}$ *Proposed by Rishabh Das*

Solution. There must be at least $\left\lfloor \frac{n}{2} \right\rfloor$ consecutive odd numbers in the n consecutive numbers.

There exists no set of 4 consecutive odd numbers that are all prime; one of them would have to be divisible by 3, and if we included 3 then the farthest we can go is $3, 5, 7$ since 9 is not prime.

Moreover, an odd number k can only be written as the sum of two primes if $k-2$ is prime, since one of the two numbers will have to be even, and we would have to write it as $(k-2)+2$.

Thus, at most 3 consecutive odd numbers can be written as the sum of two prime numbers, making n at most 7.

A construction is $4, 5, 6, 7, 8, 9, 10$, which falls out of the work done above. □

Remark. Note that after these 7, we can never reach 6 again. We would need both the Goldbach Conjecture and the Twin Prime Conjecture to prove that we can reach 5 infinitely many times.

Problem 9. **[10]** There is a unique quadruple (p, q, r, s) of prime numbers such that

$$pqrs + pqr + pq + p = 2021.$$

Compute $p + 2q + 3r + 4s$.

Answer. $\boxed{78}$ *Proposed by Rishabh Das*

Solution. Since p divides $2021 = 43 \cdot 47$, p is either 43 or 47.

If p is 43, then dividing both sides by 43 yields

$$qrs + qr + q + 1 = 47 \implies qrs + qr + q = 46.$$

Then q divides 46, so q is either 2 or 23. However, $q = 23$ is far too large, so we must have $q = 2$. Then $rs + r + 1 = 23 \implies r(s+1) = 22$, which has no solutions.

Thus, $p = 47$. Dividing both sides by 47 yields

$$qrs + qr + q + 1 = 43 \implies qrs + qr + q = 42.$$

Then q divides 42, so q is either $2, 3$, or 7. If $q = 2$, then $r(s+1) = 20$, which only has the solution $r = 5, s = 3$. If $q = 3$, then $r(s+1) = 13$, which has no solution. Finally, if $q = 7$ then $r(s+1) = 5$, which has no solution.

Thus, $p = 47, q = 2, r = 5$, and $s = 3$. The final answer is $47 + 2 \cdot 2 + 3 \cdot 5 + 4 \cdot 3 = 47 + 4 + 15 + 12 = 78$. □

Problem 10. **[Up to 10] Pick a real number from 0 to 20, inclusive.** If there are n submissions $a_1, a_2, \ldots, a_n$ and your submission is X, then your score will be $\left\lfloor 5 \cdot \max\left\{2.1 - \left|X - \sqrt{\frac{a_1^2 + a_2^2 + \ldots + a_n^2}{n}}\right|, 0\right\}\right\rfloor$.

Answer. $\boxed{\text{N.A.}}$ *Proposed by Srinath Mahankali*

The goal is to guess as close to the quadratic mean as possible. Since the quadratic mean is always at least the arithmetic mean, we would expect the answer to be larger than 10. The actual value of X was 11.50925661.

Problem 11. [**11**] Evaluate

$$1-\cfrac{1}{1+\cfrac{1}{1-\cfrac{1}{1+\cfrac{1}{1-\cfrac{1}{1+\cfrac{1}{1-\cfrac{1}{1+\cfrac{1}{1-\cfrac{1}{1+\cfrac{1}{1-\cfrac{1}{1+\cfrac{1}{1-\cfrac{1}{1+\cfrac{1}{1-\cfrac{1}{1+\cfrac{1}{1-\cfrac{1}{1+\cfrac{1}{1-\cfrac{1}{1+\cfrac{1}{1}}}}}}}}}}}}}}}}}}}}.$$

(The number 1 appears 41 times in the above fraction.)

Answer. $\boxed{\dfrac{89}{144}}$ *Proposed by Rishabh Das*

Solution. Let $f(x) = 1 - \dfrac{1}{1+\frac{1}{x}}$. We want to compute

$$f^{10}(1) = f(f(f(f(f(f(f(f(f(f(1)))))))))).$$

Let F_k denote the kth Fibonacci number. We claim

$$f\left(\frac{F_k}{F_{k+1}}\right) = \frac{F_{k+1}}{F_{k+2}}.$$

Indeed,

$$f\left(\frac{F_k}{F_{k+1}}\right) = 1 - \frac{1}{1+\frac{1}{F_k/F_{k+1}}} = 1 - \frac{1}{1+\frac{F_{k+1}}{F_k}} = 1 - \frac{1}{F_{k+2}/F_k} = 1 - \frac{F_k}{F_{k+2}} = \frac{F_{k+1}}{F_{k+2}}.$$

Then

$$f^{10}(1) = f^{10}\left(\frac{F_1}{F_2}\right) = \frac{F_{11}}{F_{12}} = \frac{89}{144}.$$

□

Problem 12. [**11**] Mr. Sterr rolls three six-sided dice. He sums the results of the first two dice, and multiplies that by the result of the third die. What is the probability that the number he obtains is a multiple of 8?

Answer. $\boxed{\dfrac{17}{72}}$ *Proposed by Theo Schiminovich*

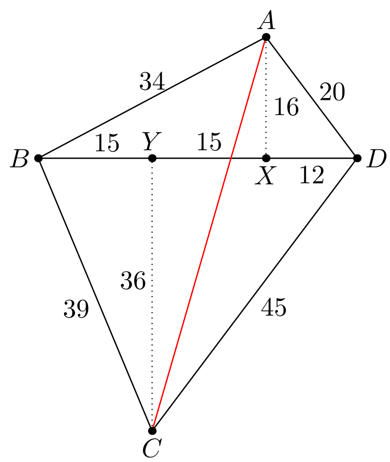
Solution. We do casework on the sum of the first two dice.
There is a $\frac{4}{36}$ chance that the sum of the first two dice is 8, and then no matter the result of the last die, the resulting number is a multiple of 8. This case gives $\frac{24}{216}$.

There is a $\frac{3+1}{36}$ chance that the sum of the first two dice is 4 or 12 and then a $\frac{3}{6}$ chance the last die is even. This case gives $\frac{18}{216}$.
There is a $\frac{1+5+3}{36}$ chance that the sum of the first dice is $2, 6$, or 10 and then a $\frac{1}{6}$ chance the last die is 4. This case gives $\frac{9}{216}$.
Overall, the probability is $\frac{24+18+9}{216} = \frac{51}{216} = \frac{17}{72}$. □

Problem 13. **[12]** Let $ABCD$ be a convex quadrilateral satisfying $AB = 34$, $BC = 39$, $CD = 45$, $DA = 20$, and $BD = 42$. What is AC^2?

Answer. $\boxed{2929}$ *Proposed by Srinath Mahankali*

Solution. Let X and Y be the feet from A and C to BD, respectively.

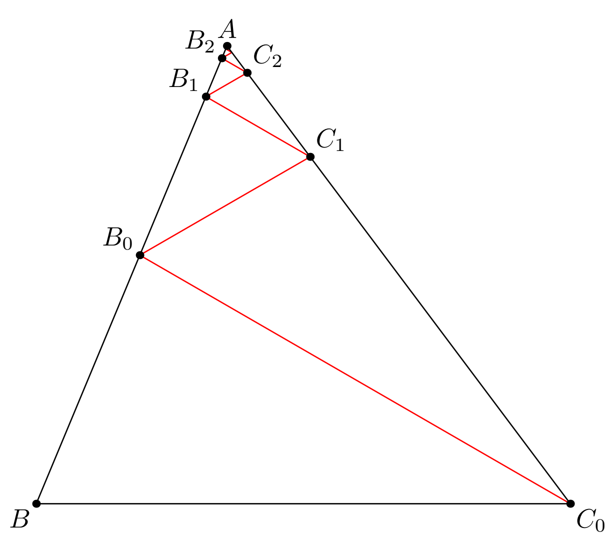


By the fact that $\triangle BCD$ is a $13-14-15$ triangle scaled up by a factor of 3 (or by using Heron's formula to find the area, and then find CY) we see $CY = 12 \cdot 3 = 36$. This gives $BY = 15$ and $YD = 27$, by either the Pythagorean theorem or by noting well-known right-triangles.

We can use a similar process to find AX, BX, and DX. We can either use Heron's formula to find the area of the triangle and then get AX, or note that the triangle is just putting a $16-30-34$ triangle and a $12-16-20$ triangle together. We get $AX = 16, BX = 30$, and $XD = 12$.

Then $XY = 15$, and then the vertical distance between A and C is $16 + 36 = 52$. Then $AC^2 = 15^2 + 52^2 = 225 + 2704 = 2929$. □

Problem 14. **[12]** In $\triangle ABC_0$, $AB = 13, BC_0 = 14$, and $AC_0 = 15$. A point B_0 is chosen on segment AB such that $\angle B_0C_0B = 30°$. For all $n \geq 1$, C_n and B_n are chosen on segments AC and AB, respectively, such that $\angle C_{n-1}B_{n-1}C_n = 60°$ and $\angle B_{n-1}C_nB_n = 60°$.



Compute

$$\sum_{i=0}^{\infty} C_iB_i + B_iC_{i+1},$$

the length of the red path.

Answer. $\boxed{24}$ *Proposed by Rishabh Das*

Solution. Note that C_iB_i and B_iC_{i+1} are always twice the length of their projection onto the A-altitude. Thus, the answer is just twice the A-altitude, which is $2 \cdot 12 = 24$. □

Problem 15. **[13]** A word is called semi-palindromic if the word and the word written in reverse have at least 50% of the corresponding letters in common. For example, $ABWXYZBA$ and $ABCXYCBA$ are semi-palindromic, but $AWXYZPQA$ is not. How many permutations of the word $MATHTEAM$ are semi-palindromic?

Answer. $\boxed{336}$ *Proposed by Theo Schiminovich*

Solution. The word $MATHTEAM$ contains 2 Ms, 2 As, 2 Ts, 1 H, and 1 E. Thus, we require either 4 or 6 letters in the permutation and its reverse to be the same.

If we have 6, then H and E must be in opposing positions in the string, and the rest of the common letters must be in opposing pairs. There are $4 \cdot 2$ ways to choose where H and E go, and then 3! ways to choose the rest, resulting in 48 total permutations.

Otherwise, we have 4 letters in the permutation and its reverse to be in common. There are $\binom{3}{2}$ ways to choose which letters are in common, and $4 \cdot 3$ ways to choose where they go. There are $\left(\binom{4}{2} - 2\right) \cdot 2 = 8$ ways to choose the rest of the letters. In total, there are 288 total permutations that work in this case.

Overall, there are 336 permutations that work. □

Problem 16. **[13]** How many sequences of length 7 consisting of the letters A, B, C, and D have the same number of As and Bs?

Answer. $\boxed{3432}$ *Proposed by Max Vaysburd*

Solution 1. Suppose there are k As and k Bs, where $0 \leq k \leq 3$. Then there are $\binom{7}{2k}$ ways to choose where the As and Bs go, $\binom{2k}{k}$ ways to choose which of these are As and which of these are Bs, and 2^{7-2k} ways to choose which of the remaining letters are C and which are D. Thus, the answer would be

$$\sum_{k=0}^{3} \binom{7}{2k}\binom{2k}{k} 2^{7-2k},$$

which we compute as $1 \cdot 1 \cdot 128 + 21 \cdot 2 \cdot 32 + 35 \cdot 6 \cdot 8 + 7 \cdot 20 \cdot 2 = 128 + 1344 + 1680 + 280 = 3432$. □

Solution 2. We use a generating function. For each spot in the sequence, represent A by x, B by $\frac{1}{x}$, and both C and D by 1. Then we want the constant in $\left(x + 2 + \frac{1}{x}\right)^7$, which is equal to the x^7 coefficient of

$$(x^2 + 2x + 1)^7 = (x + 1)^{14},$$

which is $\binom{14}{7} = 3432$. □

Solution 3. Make a 2×7 table, and select an equal number of cells from both rows. Construct a string as follows:

- if cell $(i, 1)$ was selected but not cell $(i, 2)$, then let the ith character be A;
- if cell $(i, 2)$ was selected but not cell $(i, 1)$, then let the ith character be B;
- if both cells $(i, 1)$ and $(i, 2)$ were selected, then let the ith character be C; and
- if neither cells $(i, 1)$ nor $(i, 2)$ were selected, then let the ith character be D.

Because we selected an equal number of cells from each row, there are an equal number of As and Bs in the resulting string. Conversely, from any string we can construct a valid selection of cells. Thus, we have formed a bijection.

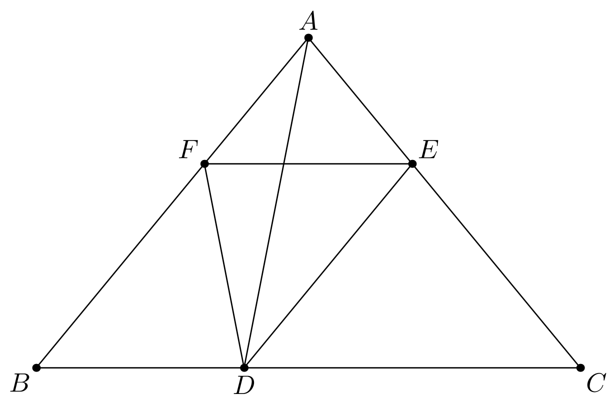
X	X			X	X	
X		X		X		X
C	A	B	D	C	A	B

Color 7 of the 14 cells green. If a cell in the top row is green, then select it, and if a cell in the bottom row is not green, then select that one. This results in an equal number of cells from each row being selected as well, and is also reversible. Thus, the answer to the original problem is $\binom{14}{7}$, since we had established a bijection. $\square$

Problem 17. [**14**] Triangle ABC has $AB = AC$. Points D, E, and F are picked on segments BC, CA, and AB, respectively, such that

$$\triangle ABC \sim \triangle AFE \sim \triangle FED \sim \triangle DAC.$$

If $CD = 1$, compute BD.



Answer. $\boxed{\dfrac{-1+\sqrt{5}}{2}}$ *Proposed by Rishabh Das*

Solution. Suppose that in all of the similar triangles, the ratio of the legs to the base is a.

Note that $\angle BDF = 180^\circ - 2\angle ACB = \angle BAC$, so $\triangle DBF$ is similar to the four triangles listed in the problem

By our definition of a, $DE = EC = a$, $EF = FD = BD = a^2$, and $AE = a^3$, so $AC = a + a^3 = \frac{CD}{a} = \frac{1}{a}$. Thus, $a^4 + a^2 - 1 = 0$, so $a^2 = \frac{-1+\sqrt{5}}{2}$. Since $BD = a^2$, this is the answer. $\square$

Problem 18. [**14**] There are two armies with 10 soldiers each. Every battle, one soldier dies, where the soldier is selected at random between all the alive soldiers. The war ends once one army has no soldiers left. Compute the expected number of soldiers left in the winning army.

Answer. $\boxed{\dfrac{20}{11}}$ *Proposed by Max Vaysburd*

Solution 1. Label the armies A and B. Notice that the process is the same as choosing a random permutation of 10 As and 10 Bs, and finding the expected value of the length of the contiguous substring at the end. For example, in the sequence

$$AABBBABABAAAABBAABBB,$$

army B wins with 3 soldiers in the end, since the last three letters of the sequence are all B.

Without loss of generality, say that army A wins the war. Then in our sequence, A will be the last letter in the sequence. Of the remaining 19 letters, 10 of them will be B and 9 of them will be A. Thus, we need to distribute

9 As into 11 slots: before the first B, between the first and second Bs, between the second and third Bs, ..., between the ninth and tenth Bs, and after the tenth B. Since each A has an equal probability of being in each slot, we expect $\frac{9}{11}$ of them to end up in the last slot. Adding the the last soldier at the end, we expect army A to win with $\frac{20}{11}$ soldiers. □

Solution 2. We define the sequences similar to the above solution, and again assume that A wins the army, with k soldiers. Then right before these k As, there must be a B. Before this, there are $10-k$ As and 9 Bs, which can be arranged in $\binom{19-k}{9}$ ways. Thus, we want to compute the sum

$$\frac{\sum_{k=1}^{10} k\binom{19-k}{9}}{\binom{19}{9}},$$

since there are $\binom{19}{9}$ total ways to arrange the letters such that A is the last letter.

We focus on computing the numerator of this expression. It is equal to

$$\sum_{k=1}^{10}\left(\sum_{m=1}^{k}\binom{19-m}{9}\right).$$

By the Hockey-Stick Identity, this is equal to

$$\sum_{k=1}^{10}\binom{20-k}{10}=\binom{20}{11}=\binom{20}{9}.$$

Then the fraction is equal to

$$\frac{\binom{20}{9}}{\binom{19}{9}}=\frac{\frac{20!}{9!\cdot 11!}}{\frac{19!}{9!\cdot 10!}}=\frac{20}{11}.$$

□

Problem 19. **[15]** A function f whose domain is the positive integers satisfies

$$f(n)=\begin{cases}1 & \text{if } n=1,\\ 3f(n/2) & \text{if } n \text{ is even, and}\\ 2f(n-1) & \text{if } n \text{ is odd and larger than } 1.\end{cases}$$

How many values less than 2021 does $f(x)$ take on over all $x\in\mathbb{N}$?

Answer. $\boxed{21}$ *Proposed by Theo Schiminovich*

Solution. We write a sequence of integers, where we subtract one at an odd number, and divide by two at an even number, until we get 1. For example, starting at 19:

$$19\longrightarrow 18\longrightarrow 9\longrightarrow 8\longrightarrow 4\longrightarrow 2\longrightarrow 1.$$

We claim that for any integer n, we use at least as many "divide by two" operations as "subtract one" operations. The reason for this is because every time we use a "subtract one" operation, it will always be immediately followed by a "divide by two" operation. Thus, the prime factorization of $f(n)$ will always have at least as many powers of 3 as powers of 2. For example:

$$19\xrightarrow{\times 2}18\xrightarrow{\times 3}9\xrightarrow{\times 2}8\xrightarrow{\times 3}4\xrightarrow{\times 3}2\xrightarrow{\times 3}1,$$

so

$$f(19)=2\cdot f(18)=2\cdot 3\cdot f(9)=2^2\cdot 3\cdot f(8)=2^2\cdot 3^2\cdot f(4)=2^2\cdot 3^3\cdot f(2)=2^2\cdot 3^4\cdot f(1)=2^2\cdot 3^4.$$

Thus, the range of f consists of numbers of the form 2^a3^b, where $a \le b$. We now claim all such numbers are in the range of f. Take $n = 2^{b-a}(2^{a+1}-1)$. Then

$$f(n) = 3^{b-a}f(2^{a+1}-1) = 2\cdot 3^{b-a}\cdot f(2^{a+1}-2) = 2\cdot 3^{b-a+1}\cdot f(2^a-1) = \cdots = 2^a\cdot 3^{b-1}\cdot f(2) = 2^a\cdot 3^b\cdot f(1) = 2^a3^b,$$

so $2^a\cdot 3^b$ where $a \le b$ is in the range of f.

Now we want to calculate how many numbers at most 2021 can be written as $2^a\cdot 3^b$ where $a \le b$. Let $b = a + c$, where $a \ge 0$. Then

$$2^a3^b = 2^a3^{a+c} = 3^c6^a,$$

so we just want the integers at most 2021 that can be written as a product of a power of 3 and a power of 6.
When the power of 6 is 1, the power of 3 is anything from 1 to 729, giving 7 numbers.
When the power of 6 is 6, the power of 3 is anything from 1 to 243, giving 6 numbers.
When the power of 6 is 36, the power of 3 is anything from 1 to 27, giving 4 numbers.
When the power of 6 is 216, the power of 3 is anything from 1 to 9, giving 3 numbers.
When the power of 6 is 1296, the power of 3 must be 1, giving 1 number.
No larger powers of 6 can work, so the answer is $7 + 6 + 4 + 3 + 1 = 21$. □

Problem 20. [Up to 64] Welcome to **USAYNO**!
Instructions: Submit a string of 6 *letters corresponding to each statement: put T if you think the statement is true, F if you think it is false, and X if you do not wish to answer. You will receive* 2^n *points for n correct answers, but you will receive* 0 *if any of the questions you choose to answer is answered incorrectly. Note that this means if you submit "XXXXXX" you will get one point.*
(1) There exists a subset S of $\{0, 1, 2, 3, 4, 5, 6, 7, 8, 9\}$ of size 4 such that the sum of the elements of any nonempty subset of S is not a multiple of 10.
(2) There exists a positive integer N such that for any integer $n \ge N$, a cube can be filled in with n non-overlapping cubes, not necessarily of the same size.
(3) There is an angle α such that there is a nonempty but finite collection of triangles which are not similar to each other, have integer side lengths, and have α as one of its angles.
(4) For any connected graph, there is always a *walk* which visits each edge exactly twice. (A walk on a graph is a sequence of moves from a vertex to an adjacent vertex.)
(5) In tetrahedron $ABCD$, the inradius of each face is equal. Then the tetrahedron must be isosceles, i.e. $AB = CD, BC = DA$, and $AC = BD$.
(6) There exists a function $f : \mathbb{N} \to \mathbb{N}$ that is **strictly increasing** such that $f(f(n)) = n^2$ for all $n \in \mathbb{N}$.

Answer. TTFTFT *Proposed by Rishabh Das, Srinath Mahankali, and Max Vaysburd*

Solution. The answer is TTFTFT.

(1) Take $S = \{2, 3, 4, 9\}$, for example.

(2) Call an integer n *good* if we can fill a cube with exactly n non-overlapping cubes. Note that if n is good, then take one of the n cubes and split it into 8 cubes. This means that $n + 7$ is good. Similarly, $n + 26$ is good, since we can split a cube into 27 cubes.
Now, since 1 is good, we can construct $1 + 7k + 26\ell$ for any $k, \ell \ge 0$. Since $7k + 26\ell$ takes on all values at least $(7 - 1)(26 - 1) = 150$ by the Chicken McNugget Theorem, $1 + 7k + 26\ell$ takes on all values at least 151. Thus, we can take $N = 151$.

(3) Note that if the condition were to be true, we would need $\cos\alpha$ to be rational. Let $\cos\alpha = k$, a rational number between -1 and 1, exclusive.
By the Law of Cosines,

$$c^2 = a^2 + b^2 - 2abk \implies \left(\frac{a}{c}\right)^2 + \left(\frac{b}{c}\right)^2 - 2\left(\frac{a}{c}\right)\left(\frac{b}{c}\right)k = 1.$$

Letting $x = \frac{a}{c}$ and $y = \frac{b}{c}$, $x^2 + y^2 - 2xyk = 1$. We will show that there are infinitely many rational points on this ellipse, which would then show, after scaling each such point appropriately, that there are infinitely many nonsimilar triangles containing α as an angle.

Note that $(1,0)$ lies on the ellipse. Take a line through $(1,0)$ with rational slope, $y = mx - m$. Replacing this in the equation of our ellipse gives a quadratic equation in x with rational coefficients. However, we already know that 1 is a root of this quadratic, so by Vieta's formulas there exists another rational root of this quadratic. If the x-coordinate is rational, from $y = mx - m$ we see the y-coordinate is also rational, so this is a rational point. Repeating this for infinitely many values of m gives the desired claim.

(4) Draw the same graph, except draw two copies of each edge. Then the graph is connected and each vertex has even degree, so there exists a Eulerian Cycle through the vertices. Tracing this cycle on our original graph gives the desired walk.

(5) Suppose that $AB = BC = CA = BD = CD = 1$, and $AD = x$. Let $f(x)$ denote the inradius of a triangle with side lengths $1, 1$, and x. If we find an x for which $f(x) = \frac{\sqrt{3}}{6}$ and $x \neq 1$, then $AD \neq BC$ but the inradius of each face is $\frac{\sqrt{3}}{6}$.
Note $f(0) = f(2) = 0$, so as long as $x = 1$ is not a maxima of f there must be another point in the range $[0,2]$ for which $f(x) = f(1)$. We will show $f(\sqrt{2}) > f(1)$, which would imply that $x = 1$ is not a maxima, so the conclusion would follow. $\sqrt{2}$ produces a $1 - 1 - \sqrt{2}$ right triangle, which has inradius $\frac{1+1-\sqrt{2}}{2} = \frac{2-\sqrt{2}}{2}$. We can show that $\frac{2-\sqrt{2}}{2} > \frac{\sqrt{3}}{6}$ by using $\sqrt{2} < 1.42$ and $\sqrt{3} < 1.72$. Thus, there exists an $x \neq 1$ such that $f(x) = f(1)$, as desired.
In fact $x = \frac{3+\sqrt{33}}{6}$.

(6) We claim that the function

$$f(n) = \begin{cases} 1 & \text{if } n = 1, \\ 3 & \text{if } n = 2, \\ [f^{-1}(n)]^2 & \text{if } n \in \text{Im}(f), \text{ and} \\ f(n-1)+1 & \text{otherwise} \end{cases}$$

works, where f^{-1} is the inverse of f and $\text{Im}(f)$ is the range of f. Just as a demonstration, here are the first few values of $f(n)$:

n	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16
$f(n)$	1	3	4	9	10	11	12	13	16	25	36	49	64	65	66	81

The condition $f(f(n)) = n^2$ is clearly satisfied. We are left to show that this is strictly increasing. We will prove the following stronger claim by strong induction: for all positive integers at least 2, $0 < f(n) - f(n-1) < 2n$.
The base case of $n = 2$ is given to us for free.
Assume that for $2 \leq k \leq n-1$, $0 < f(k) - f(k-1) < 2k$.
Call an integer a *good* if it is in the range of f, and *bad* otherwise. If n is bad, then $f(n) - f(n-1) = 1$, so the inductive step would hold.
Otherwise, suppose n is good. Suppose t is the smallest positive integer such that $n - t$ is good. If $f(a) = n$, then from the strictly increasing induction hypothesis we must have $f(a-1) = n - t$. Then $f(n-t) = (a-1)^2$, and then $f(n-1) = (a-1)^2 + t - 1$. This means

$$f(n) - f(n-1) = [a^2] - [(a-1)^2 + t - 1] = 2a - t.$$

We know $t > 0$, so $2a - t < 2a < 2n$, so $f(n) - f(n-1) < 2n$. We are left to show $f(n) - f(n-1) > 0$. From the inductive hypothesis,

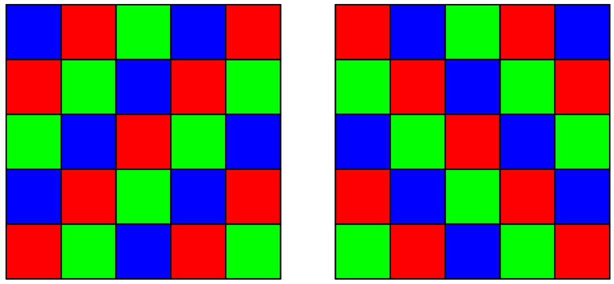
$$f(a) - f(a-1) = t < 2a,$$

so $2a - t > 0$. Thus, the inductive step holds in this case as well.
Thus, this function f satisfies the conditions of the problem. □

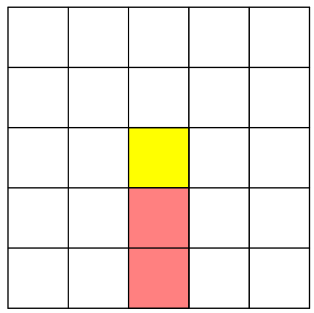
Problem 21. [16] How many ways are there to tile a 5×5 board with eight 1×3 pieces and a 1×1 piece such that no two pieces overlap?

Answer. $\boxed{2}$ *Proposed by Rishabh Das*

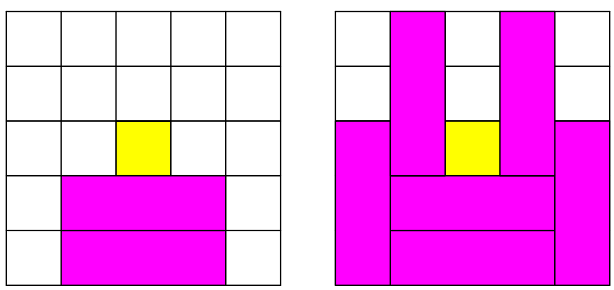
Solution. There are only two. Color the board in two ways, as follows:



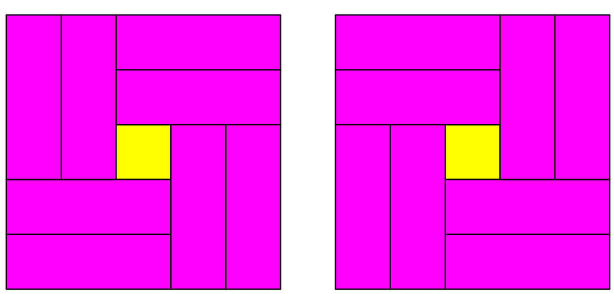
Note that each 1×3 piece will have one cell of each of the three colors. Thus, since there are 9 red cells in the first diagram on only 8 green and blue cells, the 1×1 piece must be on a red cell. Similarly, the 1×1 piece must be on a red cell in the second diagram. However, the only cell that is red on both diagrams is the center cell, so the 1×1 cell must go in the center cell.



We now focus on the two cells below the center cell. They clearly cannot be covered with vertical 1×3 pieces, so they must be covered by two distinct horizontal 1×3 pieces. Moreover, if the two 1×3 pieces are not aligned with each other, then there will be cells in the bottom two rows that are unfilled. Thus, these two 1×3 pieces will be aligned with each other. There are three possible locations for these two pieces. The first case is when they are both in the middle.



It's not too difficult to see that from the scenario on the left, we are forced into the scenario on the right, which doesn't work. Thus, we are left with the other two cases. They are symmetric, and moreover it is not too difficult to see that you are forced into one of the following two scenarios:



which both work, so the answer is 2. □

Problem 22. **[16]** A modified Pascal's Triangle follows the following rules:

- It starts with Row 0;
- Row n consists of $n + 1$ numbers, the first and last of which are both n; and

- Every other number is equal to the sum of the two numbers above it.

Here are the first few terms of the triangle:

0	Row 0
1 1	Row 1
2 2 2	Row 2
3 4 4 3	Row 3
4 7 8 7 4	Row 4

Compute the sixth number in the 16th row.

Answer. $\boxed{9828}$ *Proposed by Rishabh Das*

Solution. We claim that the kth number in the nth row is $\binom{n}{k-2} + \binom{n}{k}$. Indeed, after making this claim, it is easy to prove with induction, along with Pascal's identity. Here are a couple of ways to motivate this claim.

Let $f_n(x)$ be the generating function for the nth row. Then

$$f_{n+1}(x) = f_n(x) + xf_n(x) + x^{n+1} + 1 = (x+1)f_n(x) + (x^{n+1} + 1).$$

This recursion gives

$$f_n(x) = \sum_{m=0}^{n-1} (1+x)^m (1 + x^{n-m}).$$

The coefficient of x^{k-1} is then

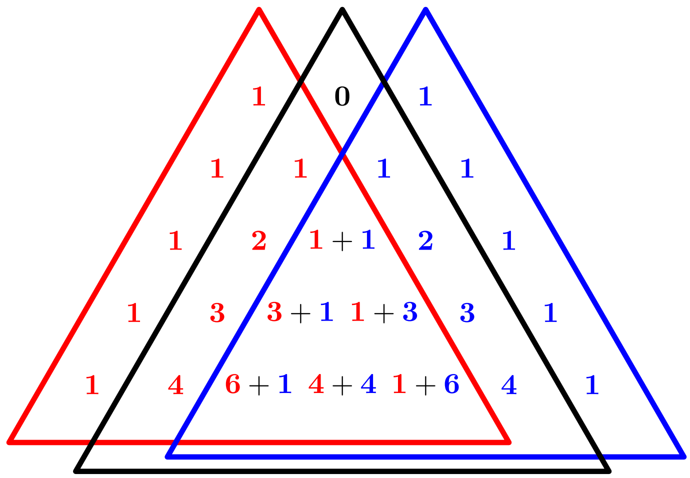
$$\sum_{m=0}^{n-1} \binom{m}{k-1} + \binom{m}{m-(n+1-k)} = \sum_{m=0}^{n-1} \binom{m}{k-1} + \binom{m}{n+1-k}.$$

By the Hockey-Stick identity, this is equal to

$$\binom{n}{k} + \binom{n}{k-2},$$

as desired.

Another way to derive the main claim is to notice that the given triangle is the superimposition of two (regular) Pascal Triangles, one shifted to the left and one shifted to the right.



The red terms (in the red triangle) form Pascal's Triangle, and the blue terms (in the blue triangle) form another Pascal's Triangle. The terms in the black triangle are the terms of the original triangle. Writing the terms in this way immediately gives $\binom{n}{k} + \binom{n}{k-2}$.

With the given numbers, the desired term is $\binom{16}{6} + \binom{16}{4} = 9828$. $\square$

Problem 23. **[17]** Let $p = 2017$ and let $f(x)$ be a polynomial with integer coefficients and degree at most $p-1$ such that $f(x+p) \equiv f(x) + px^2 \pmod{p^2}$ for all integers x. If $f(0) = 2016$, compute the sum of all possible remainders when $f(1)$ is divided by p.

Answer. $\boxed{1344}$ *Proposed by Srinath Mahankali*

Solution. Let

$$f(x) = a_0 + a_1x + a_2x^2 + \cdots + a_nx^n,$$

where $n \le p-1$. Then

$$f(x+p) = \sum_{k=0}^{n} a_k(x+p)^k \equiv \sum_{k=0}^{n} a_k(x^k + pkx^{k-1}) \pmod{p^2}$$

from the binomial theorem. Then

$$f(x+p) \equiv \left[\sum_{k=0}^{n} a_kx^n\right] + p\left[\sum_{k=1}^{n} ka_kx^{k-1}\right] \equiv f(x) + p\left[\sum_{k=1}^{n} ka_kx^{k-1}\right] \pmod{p^2}.$$

(In fact, the bracketed term might be recognized as $f'(x)$.) Thus,

$$p\left[\sum_{k=1}^{n} ka_kx^{k-1}\right] \equiv px^2 \pmod{p^2} \implies \left[\sum_{k=1}^{n} ka_kx^{k-1}\right] \equiv x^2 \pmod{p}.$$

From here it is easy to see that $a_3 \equiv \frac{1}{3} \pmod{p}$ and every other coefficient (besides potentially a_0) is 0. More formally, let the bracketed term be $g(x)$. Then $g(x) - x^2$ is a polynomial with p roots and degree at most $p-1$, so $g(x)$ must just be x^2, which gives the previous claim.

Now $f(x) \equiv \frac{1}{3}x^3 + a_0 \pmod{p}$. Looking at $f(0)$, we see $a_0 = -1$. Thus, $f(x) \equiv \frac{1}{3}x^3 - 1 \pmod{p}$. Note $\frac{1}{3} \equiv \frac{4035}{3} \equiv 1345 \pmod{2017}$, so

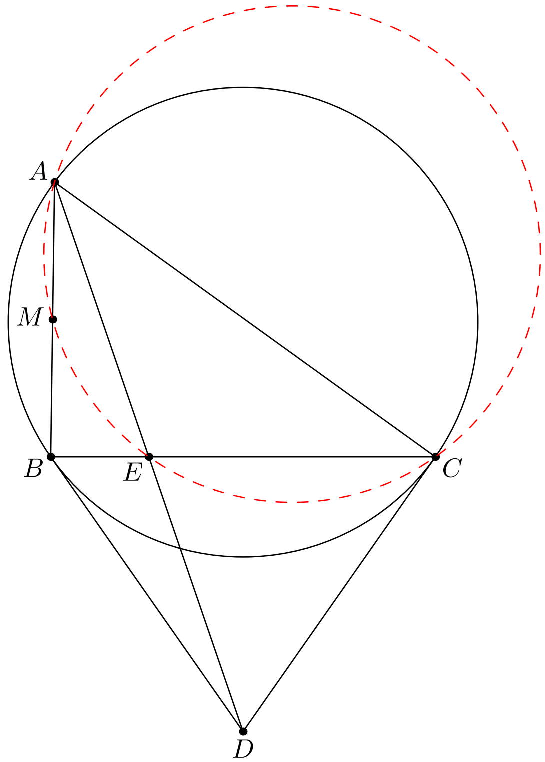
$$f(1) \equiv \frac{1}{3} - 1 \equiv 1345 - 1 \equiv 1344 \pmod{2017},$$

so 1344 is the only possible value of $f(1) \pmod{2017}$. □

Problem 24. **[17]** Let D be the intersection of tangents to the circumcircle of $\triangle ABC$ at B and C. If $AB = 20$, $AC = 21$, and the midpoint of AD lies on BC, compute BC.

Answer. $\boxed{\dfrac{29\sqrt{2}}{2}}$ *Proposed by Rishabh Das*

Solution 1. Let M denote the midpoint of AB, and E be the midpoint of AD, which lies on BC.



We claim $AMEC$ is cyclic. Note that ME is the A-midline of $\triangle ABD$, so $ME \parallel BD$. Thus

$$\angle AME = \angle ABD = \angle B + \angle A = 180^\circ - \angle C = 180^\circ - \angle ACE,$$

so $AMEC$ is indeed cyclic.

We now use power of a point with respect to B on $(AMEC)$. Then

$$BM \times BA = BE \times BC \implies \frac{c}{2} \cdot c = \frac{c^2 a}{b^2 + c^2} \cdot a \implies b^2 + c^2 = 2a^2 \implies a = \frac{\sqrt{b^2 + c^2}}{\sqrt{2}}.$$

With the given numbers, the answer is $\frac{29\sqrt{2}}{2}$. □

Solution 2. The length of the altitude from A to BC, h_a, is equal to the length of the altitude from D to BC, h_d. We know

$$\frac{1}{2} bc \sin A = \frac{a h_a}{2} \implies h_a = \frac{bc}{a} \sin \angle A.$$

Additionally,

$$\tan \angle DBC = \frac{h_d}{a/2} \implies h_d = \frac{a}{2} \tan \angle A.$$

Equating the two,

$$\frac{bc}{a} \sin \angle A = \frac{a}{2} \tan \angle A \implies \frac{bc}{a} = \frac{a}{2 \cos \angle A} \implies 2bc \cos A = a^2.$$

Then by the law of cosines

$$a^2 = b^2 + c^2 - 2bc \cos A = b^2 + c^2 - a^2 \implies a = \frac{\sqrt{b^2 + c^2}}{\sqrt{2}},$$

the same expression as before. □

Solution 3. We use barycentric coordinates. Define E as before. We know $E = (0 : b^2 : c^2)$. This means

$$D = 2 \cdot E - A = 2 \cdot (0 : b^2 : c^2) - (b^2 + c^2 : 0 : 0) = (-b^2 - c^2 : 2b^2 : 2c^2).$$

However, D is supposed to lie on the perpendicular bisector of BC. The condition for a point $(x:y:z)$ to lie on the perpendicular bisector of BC is

$$a^2(z-y)+x(c^2-b^2)=0.$$

Thus,

$$a^2(2c^2-2b^2)+(-b^2-c^2)(c^2-b^2)=0 \implies b^2+c^2=2a^2 \implies a=\frac{\sqrt{b^2+c^2}}{\sqrt{2}},$$

the same expression as before. □

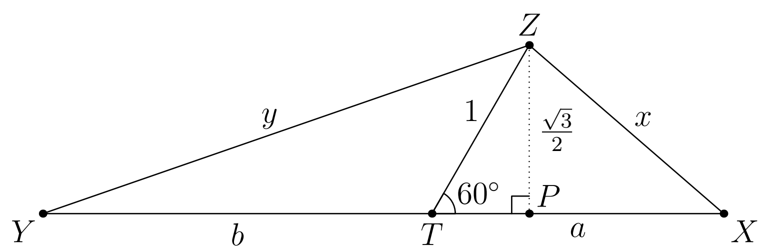
Problem 25. **[18]** If a, b, x, and y are positive real numbers satisfying

$$a^2-a+1=x^2 \text{ and}$$
$$b^2+b+1=y^2,$$

then let the maximum possible value of $\frac{a+b}{xy}$ be M. Over all quadruples (a,b,x,y) for which $\frac{a+b}{xy}=M$, suppose $a+b$ takes minimum value m. Compute $M+m$.

Answer. $\boxed{\frac{5\sqrt{3}}{3}}$ *Proposed by Rishabh Das*

Solution 1. We begin by computing M. Draw a segment XY of length $a+b$, and pick a point T on XY such that $YT=b$. Draw a segment of length 1 that forms a $60°$ angle with XY through T, TZ.



By the law of cosines, $ZX=x$ and $ZY=y$. Also note that the altitude from Z in this triangle is $\frac{\sqrt{3}}{2}$. Then

$$\frac{(a+b)\frac{\sqrt{3}}{2}}{2}=\frac{1}{2}xy\sin\angle XZY \implies \frac{a+b}{xy}=\frac{2\sin XZY}{\sqrt{3}}\le\frac{2\sqrt{3}}{3}.$$

Equality holds whenever $\angle XZY=90°$, so $M=\frac{2\sqrt{3}}{3}$.

In a right triangle, the altitude to the hypotenuse is always at most half of the length of the hypotenuse. This can be proven by inscribing the triangle in a circle. Thus,

$$XY=a+b\ge 2\cdot\frac{\sqrt{3}}{2}=\sqrt{3}$$

whenever $\frac{a+b}{xy}=M$. Equality can hold when $x=y$, so $m=\sqrt{3}$. The answer is

$$\frac{2\sqrt{3}}{3}+\sqrt{3}=\frac{5\sqrt{3}}{3}.$$

□

Solution 2. We begin by squaring the desired expression to get

$$\frac{(a+b)^2}{x^2y^2}=\frac{(a+b)^2}{(a^2-a+1)(b^2+b+1)}=\frac{16(a+b)^2}{((2a-1)^2+3)((2b+1)^2+3)},$$

where the last equality is derived by completing the square in both factors of the denominator. Now, make the substitutions $u = 2a - 1$ and $v = 2b + 1$, so we aim to minimize

$$\frac{16(a+b)^2}{((2a-1)^2+3)((2b+1)^2+3)} = \frac{4(u+v)^2}{(u^2+3)(v^2+3)}.$$

By the Cauchy-Schwarz inequality, $(u^2+3)(3+v^2) \geq 3(u+v)^2$, giving that

$$\frac{4(u+v)^2}{(u^2+3)(v^2+3)} \leq \frac{4}{3},$$

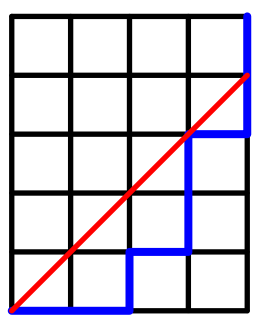
and equality is achieved when $\frac{u}{\sqrt{3}} = \frac{\sqrt{3}}{v}$, or $uv = 3$. Thus, $M^2 = \frac{4}{3}$, giving that $M = \frac{2\sqrt{3}}{3}$. Now, we must minimize $a+b$ given that $(2a-1)(2b+1) = uv = 3$. To do this, notice that it is enough to minimize $2a+2b = (2a-1)+(2b+1)$. By the AM-GM inequality, equality holds when $2a-1 = 2b+1 = \sqrt{3}$, so $a+b = \sqrt{3}$. Thus, $m = \sqrt{3}$ giving the answer of $M + m = \dfrac{5\sqrt{3}}{3}$. □

Problem 26. **[18]** Mr. Kats and Mr. Cocoros are playing a game an infinite number of times, where each game results in a win for one player and a loss for the other. Mr. Cocoros has a $\frac{3}{5}$ chance to win each game. They keep track of their respective number of wins after each game. What is the probability that there exists a point in time for which Mr. Kats has more wins than Mr. Cocoros?

Answer. $\boxed{\dfrac{2}{3}}$ *Proposed by Mario Tutuncu-Macias*

Solution. Let $p = \frac{2}{5} < \frac{1}{2}$.

The first time Mr. Kats has more wins then Mr. Cocoros, they must have $n+1$ wins and n wins, respectively, for some $n \geq 0$. We find the probability that Mr. Kats first leads in $2n+1$ games. We draw an up-right path from $(0,0)$ to $(n, n+1)$, where each time Mr. Kats wins we move up, and each time Mr. Cocoros wins we move left. We are given that this path stays below $y = x$ until it reaches (n, n), and then goes to $(n, n+1)$. Here is an example for $n = 4$.



Since the last move is fixed, we just need to get to (n, n), which is well-known to occur C_n times, the nth Catalan number. Then the probability of this happening is

$$C_n p^{n+1}(1-p)^n = p \cdot C_n \cdot (p - p^2)^n.$$

The overall probability is the sum of all of these numbers, i.e.

$$\sum_{n=0}^{\infty} p \cdot C_n \cdot (p-p^2)^n = p \cdot \sum_{n=0}^{\infty} C_n (p-p^2)^n.$$

We now use the well-known generating function

$$C(x) = \sum_{n=0}^{\infty} C_n x^n = \frac{1-\sqrt{1-4x}}{2x}.$$

Then we seek

$$pC(p-p^2) = p \cdot \frac{1-\sqrt{1-4p+4p^2}}{2(p-p^2)} = p \cdot \frac{1-(1-2p)}{2p(1-p)} = \frac{p}{1-p}.$$

For $p = \frac{2}{5}$, the answer is $\frac{2}{3}$. □

Problem 27. **[19]** How many 4-tuples of integers (a, b, c, d) are there such that $0 \leq a, b, c, d \leq 2016$,

$$a^2 + b^2 + c^2 + d^2 \equiv 1 \pmod{2017},$$

and

$$ab + cd \equiv 0 \pmod{2017}?$$

Answer. $\boxed{4064256 \text{ or } 2016^2}$ *Proposed by Srinath Mahankali*

Solution. Adding and subtracting two times the second congruence gives

$$(a+b)^2 + (c+d)^2 \equiv 1 \,(\text{mod } 2017) \;\text{ and }\; (a-b)^2 + (c-d)^2 \equiv 1 \,(\text{mod } 2017)\,.$$

Note that from the values of $a + b\,(\text{mod } 2017)$ and $a - b\,(\text{mod } 2017)$ we can uniquely recover a and b, and we get the same thing for c and d. Thus, the problem reduces to finnding the number of solutions to

$$x^2 + y^2 \equiv 1 \,(\text{mod } 2017) \;\text{ and }\; z^2 + w^2 \equiv 1 \,(\text{mod } 2017)\,.$$

We just compute the number of solutions to $x^2 + y^2 \equiv 1\,(\text{mod } 2017)$, and then square that.

Note that since $2017 \equiv 1\,(\text{mod } 4)$, there exists a k for which $k^2 \equiv -1\,(\text{mod } 2017)$. Replace y with ky. Then

$$x^2 - y^2 \equiv 1 \,(\text{mod } 2017) \implies (x+y)(x-y) \equiv 1 \,(\text{mod } 2017)\,.$$

As long as $x + y \not\equiv 0\,(\text{mod } 2017)$, we can uniquely find $x - y\,(\text{mod } 2017)$, and then recover x and y. Thus, there are 2016 solutions, making the final answer $2016^2 = 4064256$. □

Problem 28. **[19]** Akash participates in a competition with 700 students with three rounds: Algebra, Combinatorics, and Number Theory. He is 40th place in all the three rounds, and no ties occur in the three rounds. For every participant, a final score is calculated by adding the rankings from all three rounds, and an overall ranking is calculated by ranking the scores in order, with lowest score being ranked first. A contestant with a lower score will have a lower rank than a contestant with a higher score. Let M be the lowest numbered rank Akash can have and let m be the highest possible numbered rank. What is $m - M$?

Answer. $\boxed{77}$ *Proposed by Srinath Mahankali*

Solution. First of all, notice that the lowest numbered rank achievable is just first place; if the 39 people who beat Akash on each round do very poorly on the other two rounds (i.e. bottom 100), then Akash will still end up beating all of them. Thus, $M = 1$.

We seek $m - 1$, which is equal to the number of people who have a strictly lower final score than Akash. Replace 40 with n, and say k people beat Akash. We assume $k > 40$. Then we note two things:

- The maximum possible sum of scores of all people who beat Akash is $(3n-1)k$, since each of the k people who beat him will have a score of at most $3n - 1$.
- For each subject, the sum of the ranks of the people who beat Akash is at least $\frac{(k+1)(k+2)}{2} - n$, since this is the sum of the smallest k positive integers that aren't n, as rank n is taken up already.

These two combined gives

$$(3n-1)k \geq 3\left(\frac{(k+1)(k+2)}{2} - n\right).$$

We can rearrange this as

$$\begin{aligned}
3nk - k &\geq 3\left(\frac{k^2+3k+2}{2} - n\right) \\
3nk - k &\geq \frac{3}{2}k^2 + \frac{9}{2}k + 3 - 3n \\
6nk - 2k &\geq 3k^2 + 9k + 3 - 6n \\
0 &\geq 3k^2 + (11-6n)k + (6-6n).
\end{aligned}$$

This is a quadratic in k. Note that -1 is not a root, but plugging in $k = -1$ will give -2. Thus, -1 is extremely close to a root. By Vieta's the other root is approximately

$$\frac{6n-11}{3} + 1 = 2n - \frac{8}{3}.$$

(To be more rigorous, we may show that plugging in $k = -\frac{5}{3}$ gives a positive number, so there is a root between $-\frac{5}{3}$ and 1, so the other root is between $2n - \frac{8}{3}$ and $2n-2$.) Thus, the largest k that satisfies the desired inequality is the floor of this, which is $2n-3$. Thus, $k \leq 2n-3$.

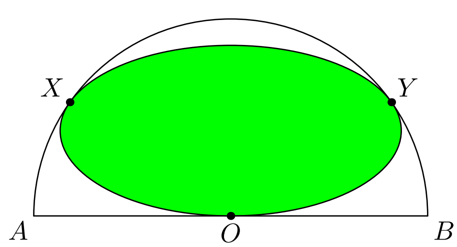
We now provide a construction for $2n-3$. First, look at the following triples.

A	C	N
1	$n+1$	$2n-3$
2	$n+2$	$2n-5$
3	$n+3$	$2n-7$
$\vdots$	$\vdots$	$\vdots$
$n-2$	$2n-2$	3
$2n-2$	$n-1$	2
$2n-3$	$n-2$	4
$2n-4$	$n-3$	6
$\vdots$	$\vdots$	$\vdots$
$n+1$	2	$2n-4$
$n-1$	1	$2n-1$

This is a list of $2n-3$ triples, and the sum of the numbers in equal triple is $3n-1$. There exists an n in the last column, however; replace that n with a 1. Then the sum of the numbers in each triple is at most $3n-1$. Thus, letting the competitors rank in the shown places for the rounds will work, showing that $k = 2n-3$ works.

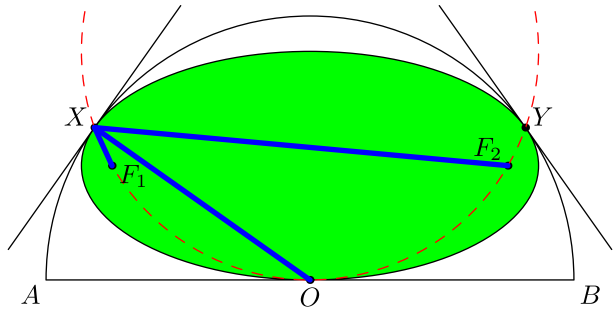
Using $n = 40$ means the answer is 77. □

Problem 29. [20] A semicircle with radius 1 is drawn, with diameter AB and center O. An ellipse is drawn tangent to AB at O, and tangent to the semicircle at 2 distinct points X and Y such that $XY \parallel AB$. Compute the maximum possible area of such an ellipse.



Answer. $\boxed{\dfrac{2\pi\sqrt{3}}{9}}$ *Proposed by Rishabh Das*

Solution. Let the foci of the ellipse be F_1 and F_2.



We claim XF_1OF_2Y is cyclic. Draw the tangent to the semicircle at X, which is also the tangent to the ellipse at X. From the reflective property of ellipses, we know that the perpendicular to this tangent through X, which is XO, is the bisector of $\angle F_1XF_2$. Since O also lies on the perpendicular bisector of F_1F_2, by the incenter-excenter lemma, O lies on the circumcircle of $\triangle F_1XF_2$. Thus, XF_1OF_2 is cyclic. Similarly, F_1OF_2Y is cyclic, so the claim is proven.

By Ptolemy's on OF_1XF_2,

$$(XF_1 + XF_2) \times OF_1 = F_1F_2 \times OX \implies 2 \cdot OF_1^2 = F_1F_2 = 2 \cdot F_1M,$$

where M is the midpoint of F_1F_2, i.e. the center of the ellipse. This means $OF_1^2 = F_1M$. Note that OF_1 is equal to the semi-major axis, and MO is equal to the semi-minor axis. Let $OF_1 = x$, so $F_1M = x^2$. Then

$$OM^2 = OF_1^2 - F_1M^2 = x^2 - x^4 \implies OM = x\sqrt{1-x^2}.$$

Thus, the area of the ellipse is

$$\pi \cdot OM \times OF_1 = \pi \cdot x^2\sqrt{1-x^2} = \pi \cdot y\sqrt{1-y},$$

where $y = x^2$. By AM-GM:

$$y^2(1-y) = 4 \cdot \frac{y}{2} \cdot \frac{y}{2} \cdot (1-y) \le 4 \cdot \left(\frac{y/2 + y/2 + (1-y)}{3}\right)^3 = \frac{4}{27} \implies y\sqrt{1-y} \le \frac{2\sqrt{3}}{9},$$

so the answer is $\frac{2\pi\sqrt{3}}{9}$. Equality holds when $y = \frac{2}{3}$, which does occur as x can range from $\frac{1}{2}$ to 1, meaning y ranges from $\frac{1}{4}$ to 1. □

Problem 30. **[20]** Given a polynomial $f(x)$ with integer coefficients and an integer n, let $\alpha(f, n)$ be the number of integers x satisfying $0 \le x < n$ and $f(x) \equiv 0 \pmod{n}$. Let $S(f) = \sum_{n=0}^{\infty} \frac{\alpha(f,n)}{n^2}$. What is

$$\frac{S(x^3 + x^2 + x)}{S(x^3 - 1)}?$$

Answer. $\boxed{\frac{89}{83}}$ *Proposed by Srinath Mahankali*

Solution. By the Chinese remainder theorem, for a fixed polynomial f, $\frac{\alpha(f,n)}{n^2}$ is a multiplicative function. Thus,

$$S(f) = \prod_p \left[\sum_{i=0}^{\infty} \frac{\alpha(f, p^i)}{p^{2i}}\right].$$

We now show

$$\alpha(x^3 + x^2 + x, p^i) = \alpha(x^3 - 1, p^i)$$

for any prime $p \neq 3$ and $i > 0$.

If $x^3+x^2+x \equiv 0 \pmod{p^i}$, then $x(x^2+x+1) \equiv 0 \pmod{p^i}$. If $p \mid x$, then $p \nmid x^2+x+1$, so either $p^i \mid x$ or $p^i \mid x^2+x+1$. This means

$$\alpha(x^3+x^2+x, p^i) = 1 + \alpha(x^2+x+1, p^i).$$

We can use a similar process for $\alpha(x^3-1, p^i)$. If $x^3-1 \equiv 0 \pmod{p^i}$, then $(x-1)(x^2+x+1) \equiv 0 \pmod{p^i}$. If $x \equiv 1 \pmod{p}$, then $x^2+x+1 \equiv 3 \not\equiv 0 \pmod{p}$, since $p \neq 3$, which means $p^i \mid x-1$ or $p^i \mid x^2+x+1$. This implies

$$\alpha(x^3-1, p^i) = 1 + \alpha(x^2+x+1, p^i).$$

Thus, $\alpha(x^3+x^2+x, p^i) = \alpha(x^3-1, p^i)$ for all primes $p \neq 3$.

We are left to compute $\alpha(x^3+x^2+x, 3^i)$ and $\alpha(x^3-1, 3^i)$. Note our reasoning for x^2+x+1 still follows, i.e.

$$\alpha(x^3+x^2+x, 3^i) = 1 + \alpha(x^2+x+1, 3^i).$$

Now

$$3^i \mid x^2+x+1 \implies 3^i \mid 4x^2+4x+4 \implies 3^i \mid (2x+1)^2+3.$$

For $i=1$, this has 1 solution. For $i>1$, we require $(2x+1)^2 \equiv -3 \pmod{9}$, which is impossible because if 3 divides a square then 9 should also divide that square.

If $3^i \mid x^3-1$, then

$$x^3-1 \equiv 0 \pmod{3} \implies x \equiv 1 \pmod{3}.$$

Then by LTE, if $i \geq 2$:

$$i \leq \nu_3(x^3-1) = 1 + \nu_3(x-1) \implies x \equiv 1 \pmod{3^{i-1}}.$$

If $i=1$ then there is one solution, and otherwise there are 3 solutions.

Finally, we want to compute

$$\prod_p \frac{\left[\sum_{i=0}^{\infty} \frac{\alpha(x^3+x^2+x, p^i)}{p^{2i}}\right]}{\left[\sum_{i=0}^{\infty} \frac{\alpha(x^3-1, p^i)}{p^{2i}}\right]} = \frac{\left[\sum_{i=0}^{\infty} \frac{\alpha(x^3+x^2+x, 3^i)}{3^{2i}}\right]}{\left[\sum_{i=0}^{\infty} \frac{\alpha(x^3-1, 3^i)}{3^{2i}}\right]} = \frac{\left[\frac{1}{3^0}+\frac{2}{3^2}+\frac{1}{3^4}+\frac{1}{3^6}+\cdots\right]}{\left[\frac{1}{3^0}+\frac{1}{3^2}+\frac{3}{3^4}+\frac{3}{3^6}+\cdots\right]}.$$

The numerator is

$$\frac{1}{1-\frac{1}{9}} + \frac{1}{9} = \frac{89}{72},$$

while the denominator is

$$3 \cdot \frac{1}{1-\frac{1}{9}} - 2 - \frac{2}{9} = \frac{83}{72}.$$

The final answer is $\dfrac{89}{83}$. □