Stuyvesant Team Contest: Problems

Spring 2021

1. The number $\overline{2111x}$ is divisible by exactly one 1-digit number d, where x is a digit. What is d?

2. **[6]** If $a + b = 2021$ and $ab + b^2 = 20210$, compute $a + b^2$.

3. **[7]** An isosceles triangle ABC has one angle equal to two times another angle. What is the sum of all possible values of $\angle A$, in degrees?

4. **[7]** How many distinct four-digit numbers are there whose digits are a permutation of the digits of 2021 (including 2021 itself)?

5. If x, y, and z are real numbers such that

$$(x + y)z = 16,$$
$$(y + z)x = 21, \text{ and}$$
$$(z + x)y = 25,$$

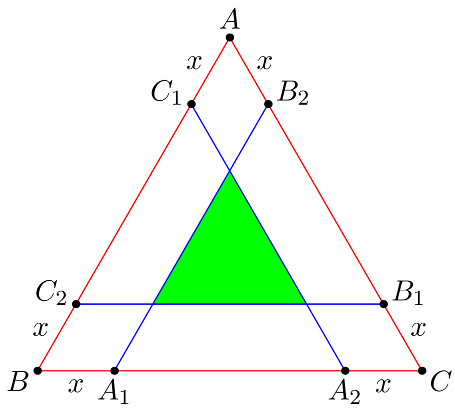
then compute $x^2 + y^2 + z^2$.

6. If a, b, c, d, and e are positive integers satisfying

$$abcd : abce : abde : acde : bcde = 1 : 2 : 3 : 4 : 5$$

then find the smallest possible value of $a + b + c + d + e$.

7. Equilateral triangle ABC has side length 1. Points A_1 and A_2 are selected on side BC such that $BA_1 = CA_2 = x$, where $x < \frac{1}{2}$. Points B_1, B_2, C_1, and C_2 are defined similarly. Lines A_1B_2, B_1C_2, and C_1A_2 define another triangle, with area $\frac{1}{4}$ of the area of ABC. Compute x.



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8. **[9]** There are n consecutive positive integers, each of which can be written as the sum of two primes. What is the maximum possible value of n?

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9. **[10]** There is a unique quadruple (p, q, r, s) of prime numbers such that

$$pqrs + pqr + pq + p = 2021.$$

Compute $p + 2q + 3r + 4s$.

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10. **[Up to 10] Pick a real number from 0 to 20, inclusive.** If there are n submissions $a_1, a_2, \ldots, a_n$ and your submission is X, then your score will be $\left\lfloor 5 \cdot \max\left\{2.1 - \left|X - \sqrt{\frac{a_1^2 + a_2^2 + \ldots + a_n^2}{n}}\right|, 0\right\}\right\rfloor$.

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11. Evaluate

$$1-\cfrac{1}{1+\cfrac{1}{1-\cfrac{1}{1+\cfrac{1}{1-\cfrac{1}{1+\cfrac{1}{1-\cfrac{1}{1+\cfrac{1}{1-\cfrac{1}{1+\cfrac{1}{1-\cfrac{1}{1+\cfrac{1}{1-\cfrac{1}{1+\cfrac{1}{1-\cfrac{1}{1+\cfrac{1}{1-\cfrac{1}{1+\cfrac{1}{1-\cfrac{1}{1+\cfrac{1}{1}}}}}}}}}}}}}}}}}}}}.$$

(The number 1 appears 41 times in the above fraction.)

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12. **[11]** Mr. Sterr rolls three six-sided dice. He sums the results of the first two dice, and multiplies that by the result of the third die. What is the probability that the number he obtains is a multiple of 8?

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13. **[12]** Let $ABCD$ be a convex quadrilateral satisfying $AB = 34$, $BC = 39$, $CD = 45$, $DA = 20$, and $BD = 42$. What is AC^2?

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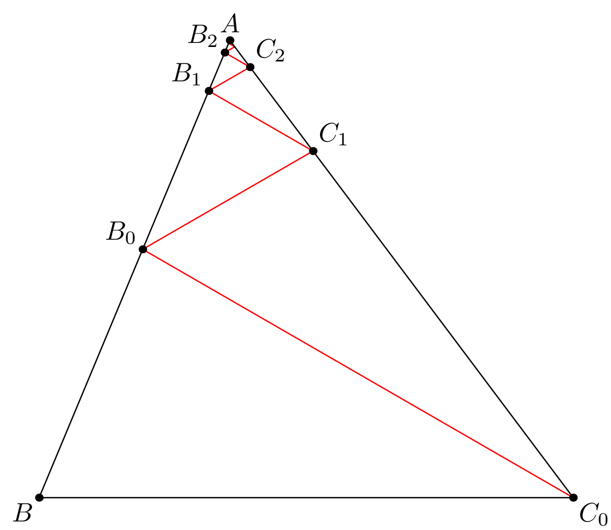
14. **[12]** In $\triangle ABC_0$, $AB = 13, BC_0 = 14$, and $AC_0 = 15$. A point B_0 is chosen on segment AB such that $\angle B_0C_0B = 30°$. For all $n \geq 1$, C_n and B_n are chosen on segments AC and AB, respectively, such that $\angle C_{n-1}B_{n-1}C_n = 60°$ and $\angle B_{n-1}C_nB_n = 60°$.



Compute

$$\sum_{i=0}^{\infty} C_iB_i + B_iC_{i+1},$$

the length of the red path.

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15. **[13]** A word is called semi-palindromic if the word and the word written in reverse have at least 50% of the corresponding letters in common. For example, $ABWXYZBA$ and $ABCXYCBA$ are semi-palindromic, but $AWXYZPQA$ is not. How many permutations of the word $MATHTEAM$ are semi-palindromic?

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16. **[13]** How many sequences of length 7 consisting of the letters A, B, C, and D have the same number of As and Bs?

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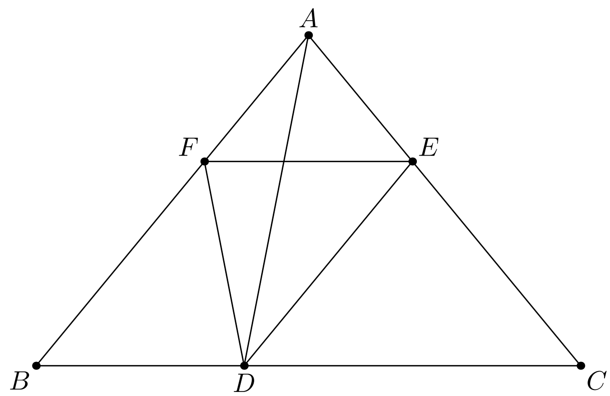
17. **[14]** Triangle ABC has $AB = AC$. Points D, E, and F are picked on segments BC, CA, and AB, respectively, such that

$$\triangle ABC \sim \triangle AFE \sim \triangle FED \sim \triangle DAC.$$

If $CD = 1$, compute BD.



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18. **[14]** There are two armies with 10 soldiers each. Every battle, one soldier dies, where the soldier is selected at random between all the alive soldiers. The war ends once one army has no soldiers left. Compute the expected number of soldiers left in the winning army.

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19. **[15]** A function f whose domain is the positive integers satisfies

$$f(n) = \begin{cases} 1 & \text{if } n = 1, \\ 3f(n/2) & \text{if } n \text{ is even, and} \\ 2f(n-1) & \text{if } n \text{ is odd and larger than 1.} \end{cases}$$

How many values less than 2021 does $f(x)$ take on over all $x \in \mathbb{N}$?

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20. **[Up to 64]** Welcome to **USAYNO**!

Instructions: Submit a string of 6 letters corresponding to each statement: put T if you think the statement is true, F if you think it is false, and X if you do not wish to answer. You will receive 2^n points for n correct answers, but you will receive 0 if any of the questions you choose to answer is answered incorrectly. Note that this means if you submit "XXXXXX" you will get one point.

(1) There exists a subset S of $\{0,1,2,3,4,5,6,7,8,9\}$ of size 4 such that the sum of the elements of any nonempty subset of S is not a multiple of 10.

(2) There exists a positive integer N such that for any integer $n \geq N$, a cube can be filled in with n non-overlapping cubes, not necessarily of the same size.

(3) There is an angle α such that there is a nonempty but finite collection of triangles which are not similar to each other, have integer side lengths, and have α as one of its angles.

(4) For any connected graph, there is always a *walk* which visits each edge exactly twice. (A walk on a graph is a sequence of moves from a vertex to an adjacent vertex.)

(5) In tetrahedron $ABCD$, the inradius of each face is equal. Then the tetrahedron must be isosceles, i.e. $AB = CD, BC = DA$, and $AC = BD$.

(6) There exists a function $f : \mathbb{N} \to \mathbb{N}$ that is **strictly increasing** such that $f(f(n)) = n^2$ for all $n \in \mathbb{N}$.

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21. **[16]** How many ways are there to tile a 5×5 board with eight 1×3 pieces and a 1×1 piece such that no two pieces overlap?

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22. **[16]** A modified Pascal's Triangle follows the following rules:

- It starts with Row 0;
- Row n consists of $n + 1$ numbers, the first and last of which are both n; and
- Every other number is equal to the sum of the two numbers above it.

Here are the first few terms of the triangle:

$$\begin{array}{ccccccccc|l} &&&&0&&&&& \text{Row } 0\\ &&&1&&1&&&& \text{Row } 1\\ &&2&&2&&2&&& \text{Row } 2\\ &3&&4&&4&&3&& \text{Row } 3\\ 4&&7&&8&&7&&4& \text{Row } 4 \end{array}$$

Compute the sixth number in the 16th row.

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23. **[17]** Let $p = 2017$ and let $f(x)$ be a polynomial with integer coefficients and degree at most $p - 1$ such that $f(x+p) \equiv f(x)+px^2 \pmod{p^2}$ for all integers x. If $f(0) = 2016$, compute the sum of all possible remainders when $f(1)$ is divided by p.

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24. **[17]** Let D be the intersection of tangents to the circumcircle of $\triangle ABC$ at B and C. If $AB = 20$, $AC = 21$, and the midpoint of AD lies on BC, compute BC.

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25. **[18]** If a, b, x, and y are positive real numbers satisfying

$$a^2 - a + 1 = x^2 \text{ and}$$
$$b^2 + b + 1 = y^2,$$

then let the maximum possible value of $\frac{a+b}{xy}$ be M. Over all quadruples (a, b, x, y) for which $\frac{a+b}{xy} = M$, suppose $a + b$ takes minimum value m. Compute $M + m$.

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26. **[18]** Mr. Kats and Mr. Cocoros are playing a game an infinite number of times, where each game results in a win for one player and a loss for the other. Mr. Cocoros has a $\frac{3}{5}$ chance to win each game. They keep track of their respective number of wins after each game. What is the probability that there exists a point in time for which Mr. Kats has more wins than Mr. Cocoros?

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27. **[19]** How many 4-tuples of integers (a, b, c, d) are there such that $0 \le a, b, c, d \le 2016$,

$$a^2 + b^2 + c^2 + d^2 \equiv 1 \pmod{2017},$$

and

$$ab + cd \equiv 0 \pmod{2017}?$$

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28. **[19]** Akash participates in a competition with 700 students with three rounds: Algebra, Combinatorics, and Number Theory. He is 40th place in all the three rounds, and no ties occur in the three rounds. For every participant, a final score is calculated by adding the rankings from all three rounds, and an overall ranking is calculated by ranking the scores in order, with lowest score being ranked first. A contestant with a lower score will have a lower rank than a contestant with a higher score. Let M be the lowest numbered rank Akash can have and let m be the highest possible numbered rank. What is $m - M$?

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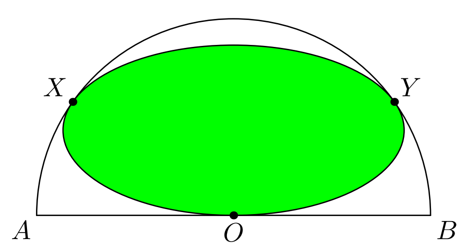
29. **[20]** A semicircle with radius 1 is drawn, with diameter AB and center O. An ellipse is drawn tangent to AB at O, and tangent to the semicircle at 2 distinct points X and Y such that $XY \parallel AB$. Compute the maximum possible area of such an ellipse.



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30. **[20]** Given a polynomial $f(x)$ with integer coefficients and an integer n, let $\alpha(f, n)$ be the number of integers x satisfying $0 \leq x < n$ and $f(x) \equiv 0 \pmod{n}$. Let $S(f) = \sum_{n=0}^{\infty} \frac{\alpha(f,n)}{n^2}$. What is

$$\frac{S(x^3 + x^2 + x)}{S(x^3 - 1)}?$$

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