2019 Spring Stuyvesant Team Contest Solutions

1. Compute
$$\frac{1}{2019^2+2019}+\frac{2019}{2019^2-2019}-\frac{2}{2019^2-1}.$$

   Solution

   Let $x = 2019$. We simplify the expression given:
$$\begin{aligned}\frac{1}{x^2+x}+\frac{x}{x^2-x}+\frac{2}{x^2-1} &= \frac{x-1}{x^3-x}+\frac{x(x+1)}{x^3-x}-\frac{2x}{x^3-x}\\ &= \frac{(x-1)+(x^2+x)-2x}{x^3-x}\\ &= \frac{x^2-1}{x^3-x}\\ &= \frac{1}{x}\end{aligned}$$

   Therefore, the answer is $\frac{1}{x}=\boxed{\frac{1}{2019}}$.

2. A mean teacher splits a group of 10 students into 3 groups. A pair of students are called "happy" if they're in the same group. What is the minimum possible number of happy pairs of students?

   Solution

   Intuitively, we would like the sizes of groups to be as close together as possible. Thus the answer should be
$$\binom{3}{2}+\binom{3}{2}+\binom{4}{2}=\boxed{12}.$$

   To prove that this is optimal, one can use a "smoothing" argument via the following claim, which tells us that making group sizes closer together decreases the number of happy pairs of students.

   Claim. *If a, b are positive integers with $a > b+1$ then*
$$\binom{a}{2}+\binom{b}{2}>\binom{a-1}{2}+\binom{b+1}{2}.$$

   This claim can be proven with simple algebraic manipulations.

3. Ethan's score on any test is at most 100. Suppose that his average after taking k tests is k, for any k less than or equal to the number of tests he takes. Compute the greatest number of tests Ethan can take.

   Solution

   Since the average of the first k test scores is k, the sum of the first k test scores is k^2. This means Ethan scored $k^2-(k-1)^2 = 2k-1$ on the k^{th} test. Since $2k-1 \le 100$ and k is an integer, we find $k \le \boxed{50}$. This is indeed attainable with scores of $1, 3, 5\ldots, 99$.

4. Compute the greatest nonnegative integer $n < 2019$ such that $2d$ is a divisor of n for all proper divisors d of n.

   Solution

   Note that $n = 0$ works. Now suppose n is positive. We claim that n must be a power of 2. Suppose n is not a power of 2. Then $n = 2^\alpha \cdot m$ for some nonnegative integer α and some odd positive integer $m > 1$. But then 2^α is a proper divisor of n, so $2^{\alpha+1}$ must also divide n. This is not possible, so n is indeed a power of 2. Powers of 2 do work, so the answer is $\boxed{1024}$.

5. Let a, b, c be positive reals such that $-a^2+b^2+c^2 : a^2-b^2+c^2 : a^2+b^2-c^2 = 1:2:3$. Compute $a:b:c$.

   Solution

   Let $S_A = -a^2+b^2+c^2$ and define S_B and S_C similarly. We have

$$\begin{array}{rrcl} & S_A : S_B : S_C & = & 1:2:3 \\ \implies & S_B+S_C : S_C+S_A : S_A+S_B & = & 2+3:3+1:1+2 \\ \implies & 2a^2 : 2b^2 : 2c^2 & = & 5:4:3 \\ \implies & a^2 : b^2 : c^2 & = & 5:4:3 \\ \implies & a:b:c & = & \boxed{\sqrt{5}:2:\sqrt{3}} \end{array}$$

6. How many of the first 20 positive integers can be expressed as both a sum and difference of two squares of integers?

   Solution

   Note that for any integer a, $a^2 \equiv 0$ or $a^2 \equiv 1 \pmod 4$. This means that for any integers x and y, $x^2+y^2 \not\equiv 3 \pmod 4$ and $x^2-y^2 \not\equiv 2 \pmod 4$. Therefore, if N is expressible as a sum and a difference of squares, $N \not\equiv 2, 3 \pmod 4$. Also, if $N \not\equiv 2 \pmod 4$, then N can indeed be written as a difference of 2 squares by factoring N into two integers of the same parity and letting these factors be $x+y$ and $x-y$. Testing for sums of squares, we find all positive integers less than or equal to 20 work, except $N = 12$. This gives us a total of $\boxed{9}$ solutions.

7. The point $(0, 0)$ is successively rotated 90° counterclockwise about each of the points

$$(1, 0), (2, 0), (3, 0), \ldots, (100, 0)$$

   in that order. Compute the area of the region above the path of the point and below the x-axis.

   Solution

   Note that the first 4 rotations follow arcs along $(0,0) \to (1,-1) \to (3,-1) \to (4,0) \to (4,0)$. Thus the path has a repeating portion every 4 rotations. The area between the x-axis and one such portion of the path is the sum of the areas of 2 quarter circles of radius 1, 1 quarter circle of radius $\sqrt{2}$, and 2 isosceles right triangles with legs of length 1. There are 25 of these repeated portions, so the answer is

$$25\left(2 \cdot \frac{1}{4} \cdot \pi \cdot 1^2 + \frac{1}{4} \cdot \pi \cdot \sqrt{2}^2 + 2 \cdot \frac{1}{2} \cdot 1 \cdot 1\right) = \boxed{25 + 25\pi}.$$

8. Compute the least positive integer n such that there exists a perfect square whose remainder when dividing by 2^n is not a perfect square.

   Solution

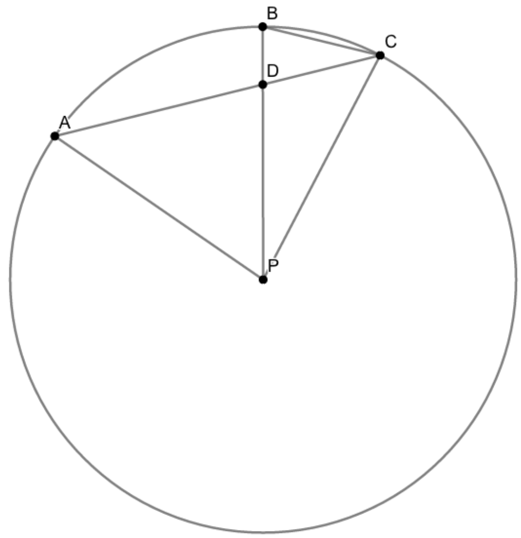
   When testing, the remainders when a perfect square is divided by 2^n are:

$$\begin{aligned} n &= 1 : 0, 1 \\ n &= 2 : 0, 1 \\ n &= 3 : 0, 1, 4 \\ n &= 4 : 0, 1, 4, 9 \end{aligned}$$

   When $n = 5$, we see that $7^2 - 32 = 17$ is not a perfect square, so we see $n = \boxed{5}$ is the minimal n.

9. Let A, B, C, and P be distinct points in the plane such that $PA = PB = PC$. Suppose that segments PB and AC intersect at D such that $BC = CD$. If $\angle APC = 60^\circ$, compute $\angle BPC$.

   Solution



Notice that we may draw a circle centered at P passing through A, B, and C, as shown above. Now, since $\triangle BCD$ and $\triangle BPC$ are both isosceles and share a base angle at B, they are similar. By inscribed arcs, $\angle APB = 2\angle ACB = 2\angle BPC$. Thus $\angle APC = 3\angle BPC$ which gives $\angle BPC = \boxed{20^\circ}$.

10. If
$$\frac{1}{K} = \frac{1}{1919} + \frac{1}{1920} + \frac{1}{1921} + \cdots + \frac{1}{2019},$$
compute $\lfloor K \rfloor$.

Solution

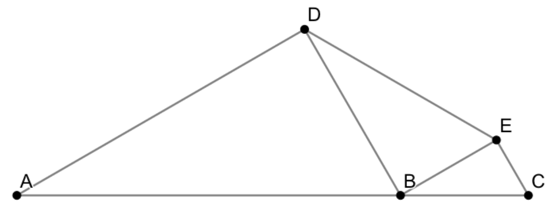
Note that
$$\frac{1}{K} < \frac{1}{1919} + \frac{1}{1919} + \cdots + \frac{1}{1919} = \frac{101}{1919} = \frac{1}{19}.$$
Moreover,
$$\frac{1}{K} > \frac{1}{2020} + \frac{1}{2020} + \cdots + \frac{1}{2020} = \frac{101}{2020} = \frac{1}{20}.$$
Therefore,
$$\frac{1}{20} < \frac{1}{K} < \frac{1}{19},$$
so $19 < K < 20$. Thus the answer is $\lfloor K \rfloor = \boxed{19}$.

11. Suppose a, b, c satisfy $abc = 1$, $a + \frac{1}{b} = \frac{1}{2}$, $b + \frac{1}{c} = \frac{3}{2}$. Compute $c + \frac{1}{a}$.

Solution

Take the product of the given expressions:
$$\begin{aligned}\left(a+\frac{1}{b}\right)\left(b+\frac{1}{c}\right)\left(c+\frac{1}{a}\right) &= abc + \frac{1}{abc} + \frac{ab}{a} + \frac{bc}{b} + \frac{ca}{c} + \frac{a}{ca} + \frac{b}{ba} + \frac{c}{bc} \\ &= 2 + a + b + c + \frac{1}{a} + \frac{1}{b} + \frac{1}{c} \\ &= 2 + \left(a+\frac{1}{b}\right) + \left(b+\frac{1}{c}\right) + \left(c+\frac{1}{a}\right).\end{aligned}$$
Let $x = c + \frac{1}{a}$. Then
$$\frac{1}{2}\cdot\frac{3}{2}\cdot x = 2 + \frac{1}{2} + \frac{3}{2} + x.$$
Therefore, $x = \boxed{-16}$.

12. In the figure below, $\triangle ADB \sim \triangle DBE \sim \triangle BEC$. If $AB = 6$ and $BC = 2$, compute DE.



Solution

Since $\triangle ADB \sim \triangle DBE$, we have that $\frac{AB}{DE} = \frac{DB}{BE}$. Since $\triangle DBE \sim \triangle BEC$, we have that $\frac{DE}{BC} = \frac{DB}{BE}$. Thus, $\frac{AB}{DE} = \frac{DE}{BC}$, so $DE^2 = AB \cdot BC = 12$, and $DE = \boxed{2\sqrt{3}}$.

13. Three runners, Mario, Max, and Maxwell, run three races. In each race there are no ties, with all other outcomes equally likely. What is the probability that Mario beats Max in the majority of the races, Max beats Maxwell in the majority of the races, and Maxwell beats Mario in the majority of the races?

Solution

Note that none of the three runners can lose two races, since otherwise they would not beat another runner in the majority of races. Similarly none of the runners can win two races. So, each runner must win one race and lose one race and thus come in second in another race. This gives two cases for the set of outcomes of the races, of which only the following works:

Mario	beats	Max	beats	Maxwell
Maxwell	beats	Mario	beats	Max
Max	beats	Maxwell	beats	Mario

So we have $3! = 6$ favorable outcomes, out of $3!^3 = 216$. Thus the answer is $\frac{6}{216} = \boxed{\frac{1}{36}}$.

14. A deck of 45 cards contains k cards labeled "k" for $k = 1, 2, \ldots, 9$. Two cards are drawn without replacement. Compute the probability that they have the same label.

Solution

There are $\binom{k}{2}$ ways to choose two of the same card with label k. There are $\binom{45}{2}$ ways to choose two cards. Thus the answer is

$$\frac{\binom{1}{2} + \binom{2}{2} + \binom{3}{2} + \cdots + \binom{9}{2}}{\binom{45}{2}} = \frac{\binom{10}{3}}{\binom{45}{2}} = \boxed{\frac{4}{33}}$$

by Hockey-Stick.

15. Kimi has n candies initially. He eats one candy and then arranges his candies into equal piles of 3. Then he eats one more candy and arranges his candies into equal piles of 5. Then he eats one more candy and arranges his candies into equal piles of 7. Then he eats one more candy and arranges his candies into equal piles of 9. Compute the least possible value of n.

Solution

For each of $k = 1, 2, 3, 4$, we have that $2k + 1 \mid n - k \implies 2k + 1 \mid 2n - 2k \implies 2k + 1 \mid 2n + 1$. Thus each of $3, 5, 7, 9$ divides $2n + 1$. Since n is positive we have $2n + 1 \geq \operatorname{lcm}(3, 5, 7, 9) = 315$. Thus $n \geq \boxed{157}$ which does work.

16. Triangle ABC has sides of length 10, 12, and x. Suppose that the A-angle bisector and the B-median are perpendicular. Compute the sum of all possible values of x.

    Solution

    Let B' be the reflection of B over the A-angle bisector, and let X be the foot from B to the A-angle bisector. Note that B' lies on AC and is also the reflection of B over X. By the givens, the line BXB' must be the B-median. Thus B' is the midpoint of AC. Therefore $AC = 2AB' = 2AB$. Since neither of 10 and 12 is twice the other, we must have one of

    $$\begin{aligned} 2 \cdot x &= 10 \\ 2 \cdot x &= 12 \\ 2 \cdot 10 &= x \\ 2 \cdot 12 &= x \end{aligned}$$

    This gives $x \in \{5, 6, 20, 24\}$. However $x = 24$ does not satisfy the triangle inequality. Thus the answer is $5 + 6 + 20 = \boxed{31}$.

17. Nancy has a circular necklace with seven beads (one for each distinct color of the rainbow). She wants to replace some (possibly none) of them with white beads, but she doesn't want any white beads to be next to each other. How many necklaces can she make like this? (If two necklaces can be rotated or flipped to match each other, they are the same necklace.)

    Solution 1

    Suppose that instead of a circular necklace, Nancy has a row of k beads. Let the number of rows she can make (with the same restrictions as in the original problem) be A_k. Note that $A_k = A_{k-1} + A_{k-2}$, by splitting into cases based on whether the last bead in the row is white or not. Also note that $A_1 = 2$ and $A_2 = 3$. Thus the sequence (A_k) is just the Fibonacci numbers. Then we have $A_4 = 8$ and $A_6 = 21$.

    Going back to the original problem, choose an arbitrary bead. If Nancy replaces this bead, she cannot replace the two adjacent ones, leaving a row of 4 beads. Similarly, if she doesn't replace this bead, she is left with a row of 6 beads to color. Thus the answer is $8 + 21 = \boxed{29}$.

    Solution 2

    Note that there can be at most 3 white beads used. If no white beads are used, Nancy can make 1 necklace. If one white bead is used, Nancy can make 7 necklaces. If two white beads are used, Nancy can make $\binom{7}{2} - 7 = 14$ necklaces (7 necklaces have two adjacent white beads). If three white beads are used, Nancy can make 7 necklaces (there is only one possible configuration, with 7 rotations). So the answer is $1 + 7 + 14 + 7 = \boxed{29}$.

18. A bag has 20 red balls and 19 green balls. Milan draws balls out of the bag until he gets a red ball. Then he passes the bag to Akash. Akash draws one ball. What is the probability that Akash draws a red ball?

    Solution

    Let us work with the $\binom{39}{19}$ possible sequences of red and green balls. We wish to count the number of such sequences in which the first red ball is followed immediately by the second red ball. This is equivalent to working with one less red ball and treating the first red ball in any resulting sequence as two red balls. Thus the answer is

    $$\frac{\binom{38}{19}}{\binom{39}{19}} = \frac{38!20!19!}{39!19!19!} = \boxed{\frac{20}{39}}.$$

19. Point C is on a circle with diameter AB. Let M be the midpoint of AC, and D be the point on ray BC such that $DM \perp AB$. If $BC = 20$ and $AD = 21$, compute CD.

    Solution 1

Construct a circle ω_A centered at A with radius 0 and a circle ω_B centered at B with radius $BC = 20$. Note that $\angle ACB = 90^\circ$, so MC is tangent to ω_B. Also MA is tangent to ω_A. Since $MA^2 = MC^2$, M has equal power with respect to ω_A and ω_B. Thus M lies on their radical axis. Now D also lies on their radical axis, since their radical axis is perpendicular to AB. So D also has equal power with respect to ω_A and ω_B. This gives $DA^2 - 0^2 = DB^2 - 20^2$. So $DB^2 = 20^2 + 21^2 = 29^2$ and $CD = BD - BC = 29 - 20 = \boxed{9}$.

Solution 2

Let $CD = x$ and $AM = MC = y$. Note the $\angle ACB = 90^\circ = \angle DCM$ and $\angle BAC = 90^\circ - \angle CBA = \angle MDC$. Thus $\triangle ABC \sim \triangle DMC$. Thus

$$\frac{CA}{CB} = \frac{CD}{CM} \implies CA \cdot CM = CD \cdot CB \implies 2y^2 = 20x.$$

Additionally, by the Pythagorean Theorem on $\triangle ACD$ we have $x^2 + 4y^2 = 21^2$. Eliminating y, we obtain

$$x^2 + 2 \cdot 20x = 21^2 \implies (x+20)^2 = 20^2 + 21^2 = 29^2 \implies x + 20 = 29 \implies x = \boxed{9}.$$

20. How many sequences of A's, B's, and C's of length 2019 have all B's followed by an A and no C's followed by an A?

Solution 1

Let us choose such a sequence backwards. The last letter can be A or C. Note that A can be preceded by A or B, B can be preceded by A or C, and C can be preceded by A or C. Thus we have 2 choices for each letter and the answer is $2^9 = \boxed{512}$.

Solution 2

Consider all sequences of length 9 composed of A's and blanks. There are 2^9 such sequences. Notice that for every occurrence of a blank in such a sequence, exactly one of B or C can be put in its spot to satisfy the rules. Namely, if a blank has an A following it, then only B can be put in its place (since no C's are followed by A's). Alternately, if a blank is followed by another blank or is the last term of the sequence, only a C can be put it it's place (since all B's must be followed by A's). Thus, number of sequences of A's, B's and C's satisfying the rules is equal to the number of ways we can make a sequence of A's and blanks, which is $2^9 = \boxed{512}$.

21. Compute the least positive integer n such that

$$\underbrace{2^{2^{\cdot^{\cdot^{\cdot^{2}}}}}}_{n\text{ 2's}} \geq \underbrace{2019^{2019^{2019^{\cdot^{\cdot^{\cdot^{2019}}}}}}}_{\text{2019 2019's}}.$$

Solution

Note that $2^{2^2} = 16 < 2019$. Thus we can increase the value of a tower of 2021 2's by replacing the top three 2's with 2019 and replacing all the other 2's with 2019's. So we must have $n \geq 2022$.

We claim that $n = 2022$ works. To see this, we will prove the following lemma.

Lemma. *Let x and y be real numbers such that $y \geq 2019$ and $x \geq 11y$. Then $2^x \geq 11 \cdot 2019^y$.*

Proof. It suffices to show the result when $x = 11y$. So, we need

$$2^{11y} \geq 11 \cdot 2019^y \iff \left(\frac{2048}{2019}\right)^y \geq 11$$

Now we have

$$\left(\frac{2048}{2019}\right)^y \geq \left(\frac{2048}{2019}\right)^{2019} = \left(1 + \frac{29}{2019}\right)^{2019} \geq 1 + 2019 \cdot \frac{29}{2019} \geq 11$$

by the Binomial Theorem, as desired. $\square$

Now note that $2^{2^{2^2}} = 65536 \geq 11 \cdot 2019$. By repeatedly applying the lemma we obtain

$$\underbrace{2^{2^{2^{\cdot^{\cdot^{\cdot^{2}}}}}}}_{2022\text{ 2's}} \geq 11 \cdot \underbrace{2019^{2019^{2019^{\cdot^{\cdot^{\cdot^{2019}}}}}}}_{2019\text{ 2019's}} \geq \underbrace{2019^{2019^{2019^{\cdot^{\cdot^{\cdot^{2019}}}}}}}_{2019\text{ 2019's}},$$

giving an answer of $n = \boxed{2022}$.

22. Compute the number of ordered triples of integers (a, b, c) for which each of

$$1 \leq a, b, c \leq 20$$

$$a \equiv b \mod c$$

$$c \equiv a \mod b$$

$$b \equiv c \mod a$$

are all true.

Solution

Assume that $a \leq b \leq c$. We will account for permutations of solutions later. Now, since $a \equiv b \mod c$ we must have $a = b$. Additionally, the other two congruences give us that $c \equiv 0 \mod a$. Thus we have $a = b \mid c$. Note that for a given value of a, b is forced and there are $\lfloor \frac{20}{a} \rfloor$ possible values of c. Summing this, we have

$$\sum_{a=1}^{20} \left\lfloor \frac{20}{a} \right\rfloor = 20 + 10 + 6 + 5 + 4 + 3 + 2 + 2 + 2 + 2 + 10 \cdot 1 = 66$$

solutions. Of these, 20 have $a = b = c$. Thus there are $66 - 20 = 46$ solutions with $a = b < c$. Accounting for permutations, we obtain an answer of $20 + 3 \cdot 46 = \boxed{158}$.

23. Compute the ordered pair of nonzero real numbers (p, q) for which the sum of the possible values of $x + y$ over solutions to the system

$$x^3 + y^3 = p$$

$$x^2 + y^2 = q$$

is 5.

Solution

Let $s = x + y$ and notice that

$$(x + y)^3 = 3(x^2 + y^2)(x + y) - 2(x^3 + y^3).$$

Rearranging and substituting, we have

$$s^3 - 3qs + 2p = 0.$$

Let this cubic in s have (not necessarily distinct) roots u, v, w. Then by Vieta, $u + v + w = 0 \neq 5$. Thus u, v, w cannot be all distinct, nor can they be all equal. Therefore we may assume $w = u \neq v$. Now, recalling the given, we have

$$2u + v = 0$$

$$u + v = 5$$

which gives $u = -5$, $v = 10$. Now by Vieta we obtain $-3q = u^2 + 2uv = -75$ and $-2p = -u^2v = -250$. Thus $(p, q) = \boxed{(125, 25)}$.

24. Compute the least positive integer n such that $a^2 + b^2 = n$ has exactly 20 integer solutions for (a, b).

Solution

Note that if (a, b) satisfies $a^2 + b^2 = n$, then so do

$$(-a, b), (a, -b), (-a, -b), (b, a), (-b, a), (b, -a), (-b, -a).$$

Thus we can put solutions to $a^2 + b^2 = n$ into groups, where a solution (a, b) is in the same group as each of the related solutions described above. Note that if a and b are distinct nonzero positive integers, we get 8 distinct solutions in the same group as (a, b). If $a = b$ or $a = 0$ or $b = 0$, then we get 4 distinct solutions in the same group as (a, b).

Note that we can have at most one group of 4 solutions, since all solutions of the form $(a, 0)$ will be in the same group, as will all solutions of the form (a, a). Furthermore it is impossible to have a solution of the form $(a, 0)$ and a solution of the form (a, a), since no positive integer is both a perfect square and twice a perfect square. It follows that if there are 20 solutions, then we must have two groups of 8 solutions and one group of 4 solutions.

Since there is a group of 4 solutions, we know that n must be either a perfect square or twice a perfect square. If n is twice a perfect square, let $n = 2k^2$. Note that for every pair (a, b) such that $a^2+b^2 = 2k^2$, a and b must have the same parity. Thus, for each pair (a, b), there is a unique pair of integers (p, q) such that $p-q = a$ and $p+q = b$. Thus, $a^2 + b^2 = 2k^2$ is equivalent to $(p-q)^2 + (p+q)^2 = 2k^2$, which reduces to $p^2 + q^2 = k^2$. This means that there are also twenty pairs (p, q) such that $p^2 + q^2 = k^2$, which contradicts the minimality of n.

Thus, n is a perfect square. Let $n = k^2$. We get four solutions from $(0, k)$, $(k, 0)$, $(-k, 0)$, and $(0, -k)$. To get 16 more solutions, we must have that $k^2 = a^2 + b^2 = c^2 + d^2$, where $a, b, c, d > 0$ and the set $\{a, b\}$ is different from the set $\{c, d\}$. By inspection, we get that $25^2 = 15^2 + 20^2 = 7^2 + 24^2$, so the answer is $25^2 = \boxed{625}$.

25. Compute the constant term in the expansion of $\left(x + \frac{1}{x} + y + \frac{1}{y}\right)^8$.

Solution 1

Let us choose 4 of the 8 factors and label them with an A, and then again (independently) choose 4 of the 8 factors and label them with an B. There are $\binom{8}{4}^2$ ways to do this. Now, in each factor, choose x, $\frac{1}{x}$, y, or $\frac{1}{y}$ corresponding to the set of labels $\{A\}$, $\{B\}$, $\{\}$, or $\{A, B\}$, respectively.

Suppose that we have k factors with a label of just A. Then we have $4-k$ factors with a label of just B, $4-(4-k) = k$ factors with a label of just B, and $4-k$ factors with no label. Thus expanding the chosen terms gives $x^k \cdot \frac{1}{x^k} \cdot y^{4-k} \cdot \frac{1}{y^{4-k}} = 1$. Since this process is reversible (we may determine the labels given choices of terms for each factor), the desired constant term is $\binom{8}{4}^2 = \boxed{4900}$.

Solution 2

Let us directly expand the given expression. To obtain a constant term, we must choose an equal number of factors of x and $\frac{1}{x}$, as well as an equal number of factors of y and $\frac{1}{y}$. Suppose that we have k factors of x. Then we must also have k factors of $\frac{1}{x}$. This also means we must have $4-k$ factors of both y and $\frac{1}{y}$. The number of ways to choose groups of $k, k, 4-k, 4-k$ out of 8 can be counted by first splitting into two groups of 4 and then splitting each group of 4 into k and $4-k$. Summing over the possible values of k, we have that the answer is

$$\sum_{k=0}^{4} \binom{8}{4}\binom{4}{k}^2 = \binom{8}{4}\sum_{k=0}^{4}\binom{4}{k}^2 = \binom{8}{4} \cdot \binom{8}{4} = \boxed{4900},$$

where we have used the fact that the sum of the squares of the 4^{th} row of Pascal's Triangle is $\binom{8}{4}$.

26. Two regular tetrahedra are given, with one inside the other and corresponding faces parallel. If the distances between corresponding faces are $\sqrt{2}$, $2\sqrt{2}$, $3\sqrt{2}$, and $4\sqrt{2}$, and the side length of the larger

tetrahedron is 1.5 times the side length of the smaller one, compute the side length of the smaller tetrahedron.

Solution

Notice that for any point inside a regular tetrahedron, the sum the distances from that point to the sides of the tetrahedron is a constant (in particular this constant is three times the volume divided by the surface area). So, let us translate the smaller tetrahedron so that it is concentric with the larger one. This will replace each given distance by the average of the given distances, $2.5\sqrt{2}$.

Next, notice that this distance is the difference between the inradii of the two tetrahedra. Let r be the smaller inradius. Then the larger inradius is $1.5r$. So we have $1.5r - r = 2.5\sqrt{2}$, which gives $r = 5\sqrt{2}$. Finally, one can compute that the desired sidelength is $\sqrt{24} \cdot r = \boxed{20\sqrt{3}}$.

27. Given positive reals x, y such that $(x - \sqrt{x^2 - 4})(y - \sqrt{y^2 - 4}) = 12 - 8\sqrt{2}$, find all possible values of

$$6xy - x^2 - y^2.$$

Solution 1

For $\sqrt{x^2 - 4}$ to be defined, we need $x \geq 2$. Similarly $y \geq 2$. Thus we may substitute $x = a + \frac{1}{a}$ and $y = b + \frac{1}{b}$ for positive reals $a, b \geq 1$. Now

$$x - \sqrt{x^2 - 4} = a + \frac{1}{a} - \sqrt{a^2 - 2 + \frac{1}{a^2}} = a + \frac{1}{a} - \sqrt{\left(a - \frac{1}{a}\right)^2} = a + \frac{1}{a} - a + \frac{1}{a} = \frac{2}{a}.$$

Similarly, $y - \sqrt{y^2 - 4} = \frac{2}{b}$. Hence we obtain

$$\frac{2}{a} \cdot \frac{2}{b} = 12 - 8\sqrt{2} \implies ab = \frac{1}{3 - 2\sqrt{2}} = 3 + 2\sqrt{2}.$$

Let $c = 3 - 2\sqrt{2}$. Note that $abc = 1$ and $c + \frac{1}{c} = 6$. Then we have

$$\begin{aligned}
6xy - x^2 - y^2 &= \left(a + \frac{1}{a}\right)\left(b + \frac{1}{b}\right)\left(c + \frac{1}{c}\right) - \left(a + \frac{1}{a}\right)^2 - \left(b + \frac{1}{b}\right)^2 - \left(c + \frac{1}{c}\right)^2 + 6^2 \\
&= abc + \frac{1}{abc} + \frac{a}{bc} + \frac{b}{ca} + \frac{c}{ab} + \frac{bc}{a} + \frac{ca}{b} + \frac{ab}{c} - a^2 - b^2 - c^2 - \frac{1}{a^2} - \frac{1}{b^2} - \frac{1}{c^2} - 2 - 2 - 2 + 36 \\
&= 2 + a^2 + b^2 + c^2 + \frac{1}{a^2} + \frac{1}{b^2} + \frac{1}{c^2} - a^2 - b^2 - c^2 - \frac{1}{a^2} - \frac{1}{b^2} - \frac{1}{c^2} + 30 \\
&= \boxed{32}
\end{aligned}$$

Solution 2

Note that

$$x - \sqrt{x^2 - 4} = \frac{4}{x + \sqrt{x^2 - 4}}.$$

Substituting, we find

$$\frac{4}{x + \sqrt{x^2 - 4}} \cdot \frac{4}{y + \sqrt{y^2 - 4}} = 12 - 8\sqrt{2}.$$

We can this rearrange to obtain

$$(x + \sqrt{x^2 - 4})(y + \sqrt{y^2 - 4}) = \frac{4 \cdot 4}{12 - 8\sqrt{2}} = \frac{4}{3 - 2\sqrt{2}} = 4(3 + 2\sqrt{2}) = 12 + 8\sqrt{2}.$$

We now have the following equations:

$$\begin{aligned}
(x - \sqrt{x^2 - 4})(y - \sqrt{y^2 - 4}) &= 12 - 8\sqrt{2}, \\
(x + \sqrt{x^2 - 4})(y + \sqrt{y^2 - 4}) &= 12 + 8\sqrt{2}.
\end{aligned}$$

Expanding both equations and adding, we obtain

$$2xy + 2\sqrt{(x^2-4)(y^2-4)} = 24.$$

Therefore,

$$\sqrt{(x^2-4)(y^2-4)} = (12 - xy).$$

Squaring and expanding both sides, we obtain

$$x^2y^2 - 4x^2 - 4y^2 + 16 = 144 - 24xy + x^2y^2.$$

Cancelling and rearranging, we obtain

$$24xy - 4x^2 - 4y^2 = 128.$$

Therefore,

$$6xy - x^2 - y^2 = \boxed{32}.$$

28. [**15**] Let $k = 2019$, $m = 2019^2$, $n = 2019 \cdot 2018$. Let $S = \{(a_1, a_2, \ldots, a_k) \mid a_1, a_2, \ldots, a_k \in \mathbb{Z}_{\geq 0}, n < a_1 + a_2 + \cdots + a_k \leq m\}$ be the set of ordered k-tuples of nonnegative integers such that their sum is greater than n and at most m. Given a k-tuple $T = (a_1, a_2, \ldots, a_k) \in S$, let $f(T) = \min\{a_1, a_2, \ldots, a_k\}$. Compute

$$\sum_{T \in S} f(T).$$

(You may express your answer in terms of common functions, but not with summation notation.)

Solution

Note that $m = n + k$.

Let $A(k, s, M)$ be the number of k-tuples $(a_1, a_2, \ldots, a_k)$ with a given sum s and with $a_i \geq M$. By stars and bars on $b_i = a_i - M$, we obtain

$$A(k, s, M) = \binom{s - kM + k - 1}{k - 1}.$$

Now the number of such k tuples with a minimum value of *exactly* M is $A(k, s, M) - A(k, s, M + 1)$ by subtracting the k-tuples where the minimum value is greater than M (and thus at least $M + 1$).

Next, we can count the desired sum by counting M for each k-tuple $T \in S$ with a minimum of M. Summing over the possible values of s and M, this gives us

$$\sum_{s=n+1}^{n+k} \sum_{M=0}^{\infty} M \cdot (A(k, s, M) - A(k, s, M + 1))$$

Notice that a given term $A(k, s, M)$, with $M > 0$, appears twice in this sum; once with a coefficient of $+M$ and once with a coefficient of $-(M - 1)$. Also, we may drop remaining terms with a coefficient of 0. Thus the sum is actually equal to

$$\sum_{s=n+1}^{n+k} \sum_{M=1}^{\infty} A(k, s, M) = \sum_{s=n+1}^{n+k} \sum_{M=1}^{\infty} \binom{s - kM + k - 1}{k - 1}$$

Now, notice that the possible values of $s - kM$ over $n + 1 \leq s \leq n + k$ and $M \geq 1$ are all integers less than or equal to $n + k - k \cdot 1 = n$ by considering remainders modulo k. Additionally, $A(k, s, M) = 0$ when $s - kM < 0$. Letting $N = s - kM$, the sum reduces to

$$\sum_{N=0}^{n} \binom{n + k - 1}{k - 1} = \binom{n + k}{k}$$

by Hockey-Stick. Substituting the given values, the answer is $\boxed{\binom{2019^2}{2019}}$.

29. In $\triangle ABC$, $AB = 13$, $BC = 14$, and $CA = 15$. Let ω be a circle centered at O tangent to segment BC at P and tangent to the circumcircle of ABC at Q on minor arc BC. Suppose that $\angle BAP = \angle QAC$. Compute $AO^2 - OP \cdot OQ$.

Solution 1

Consider an inversion with radius $\sqrt{13 \cdot 15}$ centered at A followed by a reflection over the A-angle bisector. This swaps B and C, and also swaps line BC with the circumcircle Ω of $\triangle ABC$. Since tangency is preserved, ω maps to a circle tangent to both line BC and Ω. Since $\angle BAP = \angle QAC$, P and Q are swapped. But this means that ω is fixed. Thus the power of A with respect to ω must be $13 \cdot 15 = \boxed{195}$.

Solution 2

Let Ω be the circumcircle of $\triangle ABC$, with center O'. Let M be the intersection of line PQ with major arc BAC. By Archimedes's Lemma, M is the midpoint of major arc BAC. Let N be the midpoint of minor arc BC. Let Q' be the reflection of Q over MN. Notice that Q' is on Ω. Also $\angle BAP = \angle QAC = \angle BAQ'$. So A, P, Q' are collinear. Let R be the intersection of line AP with ω other than P.

Notice that MN is a diameter of Ω and that MN is parallel to $O'P$ (both are perpendicular to BC). Now we have

$$\angle POQ = \angle MO'Q = 180^\circ - \angle QO'N = 180^\circ - 2\angle QAN = 180^\circ - \angle QAQ' = 180^\circ - \angle QAP.$$

Thus $OPAQ$ is cyclic. Since $OP = OQ$, O is the midpoint of minor arc PQ on this circle, which means AO bisects $\angle QAP$. Since AN also bisects $\angle QAP$, A, O, N are collinear.

Next, notice that O lies on the perpendicular bisector of QR and the angle bisector of $\angle QAR$. Since O is not the midpoint of arc QR on the circumcircle of QAR, $\triangle QAR$ must be isosceles. Now

$$AP^2 - OP \cdot OQ = \operatorname{Pow}_\omega A = AP \cdot AR = AP \cdot AQ.$$

As $\triangle QBA$ is similar to $\triangle CPA$, $\frac{AQ}{13} = \frac{15}{AP} \implies AP \cdot AQ = 13 \times 15 = \boxed{195}$.

30. Let P be a polynomial with rational coefficients satisfying $P(\sqrt{2} + \sqrt{3} + \sqrt{5}) = \sqrt{30}$. Compute the minimum possible degree of P.

Solution 1

By taking conjugates, we have

$$P(\varepsilon_1\sqrt{2} + \varepsilon_2\sqrt{3} + \varepsilon_3\sqrt{5}) = \varepsilon_1\varepsilon_2\varepsilon_3\sqrt{30}$$

for each of the 8 choices of $\varepsilon_1, \varepsilon_2, \varepsilon_3 = \pm 1$.

Consider the unique polynomial of degree at most 7 satisfying the given conditions. Letting $S = \{\varepsilon_1\sqrt{2} + \varepsilon_2\sqrt{3} + \varepsilon_3\sqrt{5} \mid \varepsilon_1, \varepsilon_2, \varepsilon_3 = \pm 1\}$, by Lagrange Interpolation this polynomial is

$$\sum_{s \in S} P(s) \prod_{\substack{t \in S \\ t \neq s}} \frac{x - t}{s - t}.$$

Furthermore, conjugating this polynomial just permutes the terms of the summation and the product. Thus the polynomial is fixed under conjugation, which means it has rational coefficients. Therefore P is this polynomial.

To check that P indeed has degree 7, we can check that the x^7 coefficient

$$\sum_{s \in S} P(s) \prod_{\substack{t \in S \\ t \neq s}} \frac{1}{s - t}$$

of P is nonzero. It suffices to compute one term of this sum, since all the terms are obtained from each other by conjugation. So, let $s = \sqrt{2}+\sqrt{3}+\sqrt{5}$. We have

$$P(s)\prod_{\substack{t\in S\\ t\neq s}}\frac{1}{s-t} = \frac{\sqrt{30}}{(2\sqrt{2})(2\sqrt{3})(2\sqrt{5})(2\sqrt{2}+2\sqrt{3})(2\sqrt{3}+2\sqrt{5})(2\sqrt{5}+2\sqrt{2})(2\sqrt{2}+2\sqrt{3}+2\sqrt{5})}.$$

This is a rational multiple of

$$\frac{1}{(\sqrt{2}+\sqrt{3})(\sqrt{3}+\sqrt{5})(\sqrt{5}+\sqrt{2})(\sqrt{2}+\sqrt{3}+\sqrt{5})}.$$

Rationalizing the denominator, this is a rational multiple of

$$(\sqrt{2}-\sqrt{3})(\sqrt{3}-\sqrt{5})(\sqrt{5}-\sqrt{2})(-\sqrt{2}+\sqrt{3}+\sqrt{5})(\sqrt{2}-\sqrt{3}+\sqrt{5})(\sqrt{2}+\sqrt{3}-\sqrt{5}).$$

Expanding, we obtain

$$((3-5)\sqrt{2}+(5-2)\sqrt{3}+(2-3)\sqrt{5})((-2+3+5)\sqrt{2}+(2-3+5)\sqrt{3}+(2+3-5)\sqrt{5}-2\sqrt{30})$$

When multiplied out, the rational term of this will be

$$(3-5)(-2+3+5)(2)+(5-2)(-3+5+2)(3)+(2-3)(-5+2+3)(5) = -24+36+0 = 12 \neq 0$$

Summing conjugates then gives us that the x^7 coefficient of P is 8 times a rational multiple of 12, which is nonzero as desired. Thus the answer is $\boxed{7}$.

Solution 2

Let us consider powers of $\alpha = \sqrt{2}+\sqrt{3}+\sqrt{5}$ as linear combinations of $1, \sqrt{2}, \sqrt{3}, \sqrt{5}, \sqrt{6}, \sqrt{10}, \sqrt{15}, \sqrt{30}$. Notice that in odd powers of α, we only have terms with $\sqrt{2}, \sqrt{3}, \sqrt{5}$, and $\sqrt{30}$. Furthermore these terms only appear in odd powers of α. Thus we only need to consider odd powers of α.

Next, notice that by taking conjugates, any polynomial with α as a root also has all 8 elements of $\{\varepsilon_1\sqrt{2}+\varepsilon_2\sqrt{3}+\varepsilon_3\sqrt{5} \mid \varepsilon_1, \varepsilon_2, \varepsilon_3 = \pm 1\}$ as roots. Thus the polynomial of least degree with α as a root and with rational coefficients has degree at least 8. This means that $1, \alpha, \alpha^2, \dots, \alpha^7$ are linearly independent over the rationals (otherwise there would exist a polynomial with α as a root and with rational coefficients).

Since $\alpha, \alpha^3, \alpha^5, \alpha^7$ are linearly independent and only have terms with $\sqrt{2}, \sqrt{3}, \sqrt{5}$, and $\sqrt{30}$, every linear combination of the latter 4 terms can be expressed as a linear combination of the former 4 terms. In particular, we can write $\sqrt{30}$ as a linear combination of $\alpha, \alpha^3, \alpha^5, \alpha^7$.

It remains to check that $\sqrt{30}$ is not a linear combination of $\alpha, \alpha^3, \alpha^5$. This can be done by computing these powers of α. We find that $\boxed{7}$ is indeed the answer.