2019 Spring Stuyvesant Team Contest

1. [**10**] Compute

$$\frac{1}{2019^2+2019}+\frac{2019}{2019^2-2019}-\frac{2}{2019^2-1}.$$

Team Name: ______________ Answer: ______________

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2. [**10**] A mean teacher splits a group of 10 students into 3 groups. A pair of students are called "happy" if they're in the same group. What is the minimum possible number of happy pairs of students?

Team Name: ______________ Answer: ______________

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3. [**10**] Ethan's score on any test is at most 100. Suppose that his average after taking k tests is k, for any k less than or equal to the number of tests he takes. Compute the greatest number of tests Ethan can take.

Team Name: ______________ Answer: ______________

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4. [**10**] Compute the greatest nonnegative integer $n < 2019$ such that $2d$ is a divisor of n for all proper divisors d of n.

Team Name: ______________ Answer: ______________

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5. [**10**] Let a, b, c be positive reals such that $-a^2+b^2+c^2 : a^2-b^2+c^2 : a^2+b^2-c^2 = 1:2:3$. Compute $a:b:c$.

Team Name: ______________ Answer: ______________

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6. [**11**] How many of the first 20 positive integers can be expressed as both a sum and difference of two squares of integers?

Team Name: ______________ Answer: ______________

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7. [**11**] The point $(0,0)$ is successively rotated 90° counterclockwise about each of the points

$$(1,0),(2,0),(3,0),\ldots,(100,0)$$

in that order. Compute the area of the region above the path of the point and below the x-axis.

Team Name: ______________ Answer: ______________

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8. [**11**] Compute the least positive integer n such that there exists a perfect square whose remainder when dividing by 2^n is not a perfect square.

Team Name: ______________ Answer: ______________

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9. [**11**] Let A, B, C, and P be distinct points in the plane such that $PA = PB = PC$. Suppose that segments PB and AC intersect at D such that $BC = CD$. If $\angle APC = 60^\circ$, compute $\angle BPC$.

Team Name: ______________ Answer: ______________

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10. [**11**] If

$$\frac{1}{K} = \frac{1}{1919} + \frac{1}{1920} + \frac{1}{1921} + \cdots + \frac{1}{2019},$$

compute $\lfloor K \rfloor$.

Team Name: ______________ Answer: ______________

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11. [**12**] Suppose a, b, c satisfy $abc = 1$, $a + \frac{1}{b} = \frac{1}{2}$, $b + \frac{1}{c} = \frac{3}{2}$. Compute $c + \frac{1}{a}$.

Team Name: _______________ Answer: _______________

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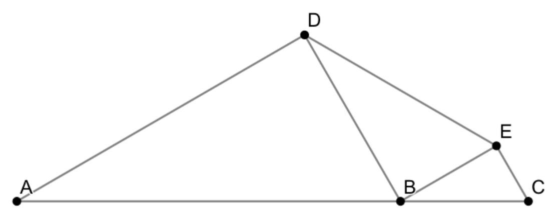
12. [**12**] In the figure below, $\triangle ADB \sim \triangle DBE \sim \triangle BEC$. If $AB = 6$ and $BC = 2$, compute DE.



Team Name: _______________ Answer: _______________

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13. [**12**] Three runners, Mario, Max, and Maxwell, run three races. In each race there are no ties, with all other outcomes equally likely. What is the probability that Mario beats Max in the majority of the races, Max beats Maxwell in the majority of the races, and Maxwell beats Mario in the majority of the races?

Team Name: _______________ Answer: _______________

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14. [**12**] A deck of 45 cards contains k cards labeled "k" for $k = 1, 2, \ldots, 9$. Two cards are drawn without replacement. Compute the probability that they have the same label.

Team Name: _______________ Answer: _______________

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15. [**12**] Kimi has n candies initially. He eats one candy and then arranges his candies into equal piles of 3. Then he eats one more candy and arranges his candies into equal piles of 5. Then he eats one more candy and arranges his candies into equal piles of 7. Then he eats one more candy and arranges his candies into equal piles of 9. Compute the least possible value of n.

Team Name: ______________ Answer: ______________

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16. [**13**] Triangle ABC has sides of length 10, 12, and x. Suppose that the A-angle bisector and the B-median are perpendicular. Compute the sum of all possible values of x.

Team Name: ______________ Answer: ______________

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17. [**13**] Nancy has a circular necklace with seven beads (one for each distinct color of the rainbow). She wants to replace some (possibly none) of them with white beads, but she doesn't want any white beads to be next to each other. How many necklaces can she make like this? (If two necklaces can be rotated or flipped to match each other, they are the same necklace.)

Team Name: ______________ Answer: ______________

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18. [**13**] A bag has 20 red balls and 19 green balls. Milan draws balls out of the bag until he gets a red ball. Then he passes the bag to Akash. Akash draws one ball. What is the probability that Akash draws a red ball?

Team Name: ______________ Answer: ______________

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19. [**13**] Point C is on a circle with diameter AB. Let M be the midpoint of AC, and D be the point on ray BC such that $DM \perp AB$. If $BC = 20$ and $AD = 21$, compute CD.

Team Name: ______________ Answer: ______________

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20. [**13**] How many sequences of A's, B's, and C's of length 9 have all B's followed by an A and no C's followed by an A?

Team Name: ______________ Answer: ______________

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21. [**14**] Compute the least positive integer n such that

$$\underbrace{2^{2^{\cdot^{\cdot^{\cdot^{2}}}}}}_{n\text{ 2's}} \geq \underbrace{2019^{2019^{2019^{\cdot^{\cdot^{\cdot^{2019}}}}}}}_{2019\text{ 2019's}}.$$

Team Name: ______________ Answer: ______________

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22. [**14**] Compute the number of ordered triples of integers (a, b, c) for which each of

$$1 \leq a, b, c \leq 20$$
$$a \equiv b \pmod c$$
$$c \equiv a \pmod b$$
$$b \equiv c \pmod a$$

are all true.

Team Name: ______________ Answer: ______________

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23. [**14**] Compute the ordered pair of nonzero real numbers (p, q) for which the sum of the possible values of $x + y$ over solutions to the system

$$x^3 + y^3 = p$$
$$x^2 + y^2 = q$$

is 5.

Team Name: ______________ Answer: ______________

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24. [**14**] Compute the least positive integer n such that $a^2 + b^2 = n$ has exactly 20 integer solutions for (a, b).

Team Name: ______________ Answer: ______________

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25. [**14**] Compute the constant term in the expansion of $\left(x + \frac{1}{x} + y + \frac{1}{y}\right)^8$.

Team Name: ______________ Answer: ______________

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26. [**15**] Two regular tetrahedra are given, with one inside the other and corresponding faces parallel. If the distances between corresponding faces are $\sqrt{2}$, $2\sqrt{2}$, $3\sqrt{2}$, and $4\sqrt{2}$, and the side length of the larger tetrahedron is 1.5 times the side length of the smaller one, compute the side length of the smaller tetrahedron.

Team Name: ______________ Answer: ______________

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27. [**15**] Given positive reals x, y such that $(x - \sqrt{x^2 - 4})(y - \sqrt{y^2 - 4}) = 12 - 8\sqrt{2}$, find all possible values of

$$6xy - x^2 - y^2.$$

Team Name: ______________ Answer: ______________

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28. [**15**] Let $k = 2019$, $m = 2019^2$, $n = 2019 \cdot 2018$. Let $S = \{(a_1, a_2, \ldots, a_k) \mid a_1, a_2, \ldots, a_k \in \mathbb{Z}_{\geq 0}, n < a_1 + a_2 + \cdots + a_k \leq m\}$ be the set of ordered k-tuples of nonnegative integers such that their sum is greater than n and at most m. Given a k-tuple $T = (a_1, a_2, \ldots, a_k) \in S$, let $f(T) = \min\{a_1, a_2, \ldots, a_k\}$. Compute

$$\sum_{T \in S} f(T).$$

(You may express your answer in terms of common functions, but not with summation notation.)

Team Name: ______________ Answer: ______________

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29. [**15**] In $\triangle ABC$, $AB = 13$, $BC = 14$, and $CA = 15$. Let ω be a circle centered at O tangent to segment BC at P and tangent to the circumcircle of ABC at Q on minor arc BC. Suppose that $\angle BAP = \angle QAC$. Compute $AO^2 - OP \cdot OQ$.

Team Name: _______________ Answer: _______________

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30. [**15**] Let P be a polynomial with rational coefficients satisfying $P(\sqrt{2} + \sqrt{3} + \sqrt{5}) = \sqrt{30}$. Compute the minimum possible degree of P.

Team Name: _______________ Answer: _______________

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