2018 Stuyvesant Team Contest Problems

1. [**5**] Compute:

$$\left(\frac{1}{6}\right)^3 + \left(\frac{1}{3}\right)^3 + \left(\frac{1}{2}\right)^3$$

Team Name: ______________ Answer: ______________

2. [**5**] How many non-congruent, non-degenerate triangles have integer side lengths and perimeter less than 8?

Team Name: ______________ Answer: ______________

3. [**5**] Compute the number of nonnegative integers n such that $3^n \geq n!$.

Note: $n!$ is defined as the product of all the positive integers less than or equal to n. An empty product is equal to 1.

Team Name: ______________ Answer: ______________

4. [**6**] Let a, b, c be reals such that $\frac{a+b}{c} = 1$ and $\frac{b+c}{a} = 2$. Compute $\frac{c+a}{b}$.

Team Name: ______________ Answer: ______________

5. [**6**] If a and b are positive integers such that $2a^2 = 3b^3$, compute the minimum possible value of $a + b$.

Team Name: ______________ Answer: ______________

6. [**6**] Three distinct lines are drawn in the plane. One pair forms an angle of $83°$. Another pair forms an angle of $97°$. The third pair forms an angle of $n°$. Compute smallest possible value of n.

Team Name: ______________ Answer: ______________

7. [**7**] How many subsets of the set $\{1, 2, 4, 8, 16, -1, -2, -4, -8, -16\}$ have a sum of 0?

Note: The empty set has a sum of 0.

Team Name: ______________ Answer: ______________

8. [**7**] How many distinct values does $\lfloor x \rfloor + \lceil x \rceil$ take for x in the interval $(-2018, 2018)$?

Team Name: ______________ Answer: ______________

9. [**7**] Stan takes a rectangular 24 by 32 sheet of paper and folds one of its corners onto the opposite corner. Compute the area of the pentagon formed by the folded paper.

Team Name: ______________ Answer: ______________

10. [**8**] Let $d(n)$ denote the number of positive divisors of n. Compute the largest integer $n < 100$ such that

$$d(d(d(n))) = d(n).$$

Team Name: ______________ Answer: ______________

11. [**8**] Compute the number of triples of positive integers (a, b, c) for which $18! \mid a \mid b \mid c \mid 20!$.

Note: We write $m \mid n$ if m is a factor of n.

Team Name: ______________ Answer: ______________

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12. [**8**] In $\triangle ABC$, $AB = 13$, $BC = 14$, and $AC = 15$. Points X, Y, and Z are the trisection points of sides BC, AC, and AB, respectively with X closer to B, Y closer to C, and Z closer to A. Let A', B', and C' be the reflections of A, B, and C over X, Y, and Z, respectively. Compute $[A'BC'AB'C]$.

Note: $[\cdots]$ denotes the area of the polygon in the brackets.

Team Name: ______________ Answer: ______________

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13. [**9**] Matthew flips 2018 fair coins. Milan flips 2017 fair coins. Compute the probability that Matthew gets more heads than Milan.

Team Name: ______________ Answer: ______________

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14. [**9**] A sequence $\{a_n\}$ is defined for positive integers n so that $a_1 = a_2 = 1$ and for all $n > 2$,

$$a_n = \frac{a_{n-1} \cdot a_{n-2}}{a_{n-1} + a_{n-2}}$$

Compute a_{12}.

Team Name: ______________ Answer: ______________

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15. [**9**] In right $\triangle ABC$, $AB = 1$, $BC = 4\sqrt{3}$, $AC = 7$. A circle is drawn through A and B intersecting segment AC at point P such that minor arc AB has measure 60°. Compute CP.

Team Name: ______________ Answer: ______________

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16. [**10**] Solve for all real x that satisfy:

$$\sqrt{x} - \sqrt[3]{x} + \sqrt[4]{x} = x^2 - x^3 + x^4$$

Team Name: ______________ Answer: ______________

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17. [**10**] Compute the prime factorization of $82^3 + 1$.

Team Name: ______________ Answer: ______________

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18. [**10**] In $\triangle ABC$, M and N are the midpoints of AB and AC. $AB = 8$ and $AC = 9$. The circumcircle of $\triangle AMN$ is tangent to BC at D. Compute AD.

Team Name: ______________ Answer: ______________

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19. [**11**] Compute the total sum of the digits of all three digit numbers with at least one digit that is a 7.

Team Name: ______________ Answer: ______________

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20. [**11**] How many proper divisors of 10! are a product of a perfect square and a perfect cube (not necessarily distinct).

Team Name: ______________ Answer: ______________

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21. [**11**] A non-degenerate triangle has sides of length $\frac{1}{x}$, 1, and x. Compute the set of possible values for its area.

Team Name: ______________ Answer: ______________

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22. [**12**] Hanna placed 2018 candies in a circle and labeled them $1, 2, 3, \dots 2018$ in order. Kimi eats candy 1, skips candy 2, eats candy 3, and continues going around the circle eating every other candy until he has eaten all of them. Compute the label of the last candy Kimi eats.

Team Name: ______________ Answer: ______________

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23. [**12**] In convex quadrilateral $ABCD$, the diagonals intersect at point P. If $\tan\angle BAC = 1$, $\tan\angle DAC = 2$, $\tan\angle DCA = 3$, and $\tan\angle BCA = 4$, compute $\tan\angle BPC$.

Team Name: ______________ Answer: ______________

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24. [**12**] $\sqrt{1}+\sqrt{2}+\sqrt{3}+\cdots+\sqrt{30}$ is a root of a polynomial with integer coefficients and degree n. Compute the minimum possible value of n.

Team Name: ______________ Answer: ______________

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