Stuyvesant Team Contest: Solutions

Spring 2022

Problem 1. **[6]** Suppose $T = TNYWR$, where T, N, Y, W, and R are distinct integers. What is T?

Proposed by Rishabh Das

Answer. $\boxed{0}$

Solution. Either $T = 0$ or $NYWR = 1$. However, there do not exist four distinct integers that multiply to 1, so this latter case is impossible, and $T = 0$.

Problem 2. **[6]** Find the smallest positive integer that is one more than a multiple of 6, and is neither a prime nor a perfect square.

Proposed by Rishabh Das

Answer. $\boxed{55}$

Solution. The numbers $1, 25$, and 49 are squares while $7, 13, 19, 31, 37$, and 43 are prime, leaving 55 as the smallest such number.

Problem 3. **[7]** What is the minimum number of coin flips to guarantee you can find a consecutive string of HH, TT, or $HTHT$ (where H represents flipping a heads, and T represents flipping a tails)?

Proposed by Jerry Liang

Answer. $\boxed{5}$

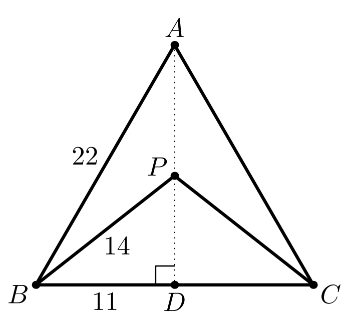
Solution. Because we must avoid HH and TT, we assume that we alternate between H's and T's. To delay starting at H for as long as possible, we should start with a T. The longest possible sequence is then $THTHT$, which has length 5.

Problem 4. **[7]** Equilateral triangle ABC has side length 22. A point P inside the triangle is chosen such that $PB = PC = 14$. Compute PA.

Proposed by Rishabh Das

Answer. $\boxed{6\sqrt{3}}$

Solution.

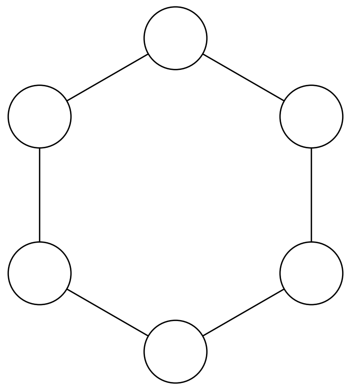


Let D be the midpoint of BC. Since $BP = PC$, P lies on the perpendicular bisector of BC. Because $BP = 14$ and $BD = 11$, we have

$$PD = \sqrt{14^2 - 11^2} = \sqrt{75} = 5\sqrt{3}.$$

Then $PA = AD - PD = 11\sqrt{3} - 5\sqrt{3} = 6\sqrt{3}$.

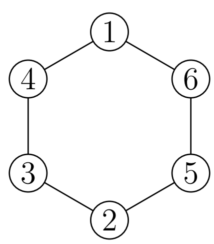
Problem 5. [**8**] Six circles are arranged into a hexagon, as shown, and the six numbers $1, 2, 3, 4, 5$, and 6 are put into these circles in some order. On each edge of the hexagon, Stanley writes down the sum of the two numbers in the circles connected to this edge, and Brian writes down the product of the two numbers in the circles connected to this edge. Stanley then notices that he wrote down only prime numbers. What is the sum of the numbers Brian wrote down?



Proposed by Rishabh Das

Answer. $\boxed{68}$

Solution. The filled in circles are below.



No two of $2, 4$, and 6 can be next to each other, since their pairwise sums are even numbers greater than 2.

Then the 3 can't be next to the 6, and the 5 can't be next to the 4. This fixes the entire grid, up to rotation and reflection.

Thus, Brian wrote down the numbers $6, 30, 10, 6, 12, 4$, making the sum 68.

Remark. Somewhat coincidentally, Brian doesn't write down *any* prime numbers.

Problem 6. [**8**] Mr. Cocoros collects dice. A die of "size n" has n faces labeled $1, 2, \ldots, n$, each equally likely to show up. Mr. Cocoros has all dice of size 4 to N for some positive integer N. Mr. Cocoros rolls all of his dice once. If the probability that Mr. Cocoros rolls at least one "1" is 75%, what is N?

Proposed by Jerry Liang

Answer. $\boxed{12}$

Solution. The probability a die of size n isn't a 1 is $\frac{n-1}{n}$. Thus, the probability that none of his dice roll a 1 is

$$\frac{3}{4}\cdot\frac{4}{5}\cdots\frac{N-1}{N}=\frac{3}{N}.$$

Thus, we need $1-\frac{3}{N}=\frac{3}{4}$, so $N=12$.

Problem 7. **[9]** The first two numbers in a list of integers are 20 and 22, respectively. If the sum of the numbers in the list is equal to the product of the numbers in the list, what's the least possible number of entries in the list?

Proposed by Rishabh Das and Josiah Moltz

Answer. $\boxed{4}$

Solution. If there were only three integers in the list, let the third one be x. Then

$$20\cdot 22\cdot x=20+22+x \implies 439x=42,$$

contradicting x being an integer. Thus, we need at least four integers.

We can construct this with the sequence $20, 22, 0, -42$, so the answer is 4.

Problem 8. **[9]** Suppose three of the vertices of a cube are $(2,4,8), (12,5,3)$, and $(11,7,14)$. Compute the volume of this cube.

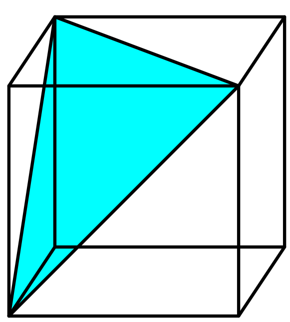
Proposed by Rishabh Das

Answer. $\boxed{189\sqrt{7}}$

Solution. Label the three points A, B, and C, in order. Then

$$\begin{aligned}AB&=\sqrt{(2-12)^2+(4-5)^2+(8-3)^2}=\sqrt{100+1+25}=\sqrt{126},\\ BC&=\sqrt{(12-11)^2+(5-7)^2+(3-14)^2}=\sqrt{1+4+121}=\sqrt{126}, \text{ and}\\ CA&=\sqrt{(11-2)^2+(7-4)^2+(14-8)^2}=\sqrt{81+9+36}=\sqrt{126}.\end{aligned}$$

This means that $\triangle ABC$ is equilateral. If three of the vertices of a cube form an equilateral triangle, then the side length of this triangle must be $\sqrt{2}$ times the side length of the cube.



Then the side length is $\sqrt{63}$, so the volume is

$$\left(\sqrt{63}\right)^3=63\sqrt{63}=189\sqrt{7}.$$

Problem 9. **[10]** Suppose $N=\underline{a}\,\underline{b}\,\underline{b}\,\underline{c}\,\underline{b}\,\underline{a}_3=\underline{a}\,\underline{b}\,\underline{b}\,\underline{a}\,\underline{c}_4$, (these are the base three and base four representations of N) where a, b, and c represent different digits. Compute N in base 5.

Proposed by Rishabh Das

Answer. $\boxed{2022_5}$

Solution. We have

$$N = \underline{a}\,\underline{b}\,\underline{b}\,\underline{c}\,\underline{b}\,\underline{a}_3 = 243a + 81b + 27b + 9c + 3b + a = 244a + 111b + 9c$$

and

$$N = \underline{a}\,\underline{b}\,\underline{b}\,\underline{a}\,\underline{c}_4 = 256a + 64b + 16b + 4a + c = 260a + 80b + c.$$

Setting these two equal to each other,

$$244a + 111b + 9c = 260a + 80b + c \implies 31b + 8c = 16a.$$

This means $8 \mid 31b$, so $b = 0$, as $b < 3$. Then $8c = 16a$, so $c = 2$ and $a = 1$. Then we can compute $N = 262 = 2022_5$.

Problem 10. [Up to 10] Pick a number between 0 **and** 1**, inclusive.** Let σ denote the standard deviation of all entries. Estimate 2σ. (Not sponsored by the way.)

If your submission is X and the actual value of 2σ is Y, you will receive

$$\max\left(\left\lfloor \frac{60}{|X-Y|+1} - 50 \right\rfloor, 0\right)$$

points.

Proposed by Rishabh Das

Answer. $\boxed{\text{N.A.}}$

The value of σ was LOREM IPSUM, making the value of 2σ LOREM IPSUM. The closest team guessed LOREM IPSUM, giving them LOREM IPSUM points.

Remark. In fact, it can be proven that twice the standard deviation is always at most the range, so 2σ is guaranteed to be in the range $[0, 1]$.

Problem 11. [11] Compute the positive integer n satisfying

$$(n-16)^2 + (n-15)^2 + \cdots + (n-1)^2 + n^2 = (n+1)^2 + (n+2)^2 + \cdots + (n+15)^2 + (n+16)^2.$$

Proposed by Rishabh Das

Answer. $\boxed{544}$

Solution. Expanding both sides,

$$17n^2 - 2\cdot(1+2+\cdots+16)n + (1^2+2^2+\cdots+16^2) = 16n^2 + 2\cdot(1+2+\cdots+16)n + (1^2+2^2+\cdots+16^2).$$

The constant terms cancel, so

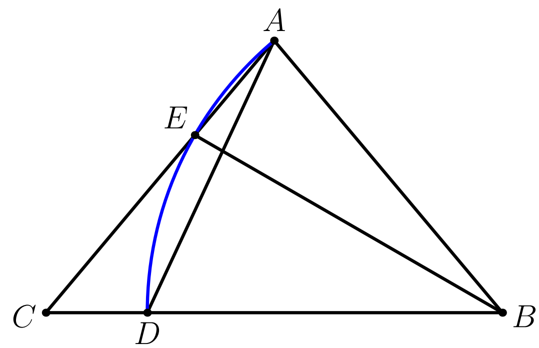
$$n^2 = 4\cdot(1+2+\cdots+16)n \implies n = 2\cdot 16\cdot 17 = 544.$$

Problem 12. [11] Let $\triangle ABC$ be an isosceles triangle with vertex A and $\angle BAC = 80^\circ$. Let D be on segment BC such that $\angle BAD = 65^\circ$, and let E be on segment AC such that $\angle EBC = 30^\circ$. Compute $\angle BED$.

Proposed by Joseph Othman

Answer. $\boxed{75^\circ}$

Solution.



Since ABC is isosceles, $\angle ACB = \angle CBA = 50^\circ$.

Focusing on triangle ABD, since $\angle DBA = 50^\circ$ and $\angle BAD = 65^\circ$, we realize $\angle ADB = 65^\circ$, meaning $AB = BD$.

Now looking at triangle ABE, since $\angle EAB = 80^\circ$ and $\angle ABE = \angle ABC - \angle EBC = 50^\circ - 30^\circ = 20^\circ$, we see that $\angle BEA = 80^\circ$, and so $AB = BE$.

Now finally we know that $\angle EBD = 30^\circ$, and so since $BE = AB = BD$ we have triangle EBD is isosceles and so $\angle BED = 75^\circ$.

Problem 13. **[12]** How many ordered triples of positive integers (a, b, c) are there such that $(a^a \times b^b \times c^c) \mid 6^6$? (We say "$x \mid y$" if x is a divisor of y.)

Proposed by Joseph Othman

Answer. $\boxed{29}$

Solution. Note that we clearly have $a, b, c \leq 6$. Clearly, none of them can be 5. Moreover, if any of them are 4, then $a^a \times b^b \times c^c$ has 8 factors of 2 in it, while 6^6 only has 6, a contradiction. Thus, $a, b, c \in \{1, 2, 3, 6\}$.

We perform casework on the number of 1s in a, b, and c.

If there are three 1's, then we have one case.

If there are two 1's, then the last number can be either $2, 3$, or 6. Accounting for the permutations, there are $3 \cdot 3 = 9$ cases.

If there is one 1, then the other two numbers must be some permutation of $(2, 2)$, $(3, 3)$, and $(2, 3)$. This gives $3 + 3 + 6 = 12$ cases accounting for permutations.

If there are no 1's, then the three numbers must be some permutation of $(2, 2, 2), (2, 2, 3)$, and $(2, 3, 3)$. This gives $1 + 3 + 3 = 7$ cases total.

Overall, there are $1 + 9 + 12 + 7 = 29$ possible ordered triples.

Problem 14. **[12]** I was born on May 10, 2004. My dad was born on July 13, 1968. Presuming both of us live until 100, for how many days will my dad be twice my age?

Proposed by Vidur Jasuja

Answer. $\boxed{365}$

Solution. I claim that for every day (excluding February 29), there exists exactly 1 year such that my dad is twice my age. On any day of the year between May 10, $(2004+n)$ and May 9, $(2005+n)$, my age will be exactly n

years old, and my dad will be exactly $n+k$ years old, where k only depends on whether or not my dad's birthday has past in this time frame. Then the ratio of my dad's age to my age will be $\frac{n+k}{n}$. Then for my dad to be twice my age, we need $k = n$. Then for each of the 365 days of a non-leap year, my dad is exactly twice my age in exactly one year.

It remains to check February 29. Note that the first leap year we are both alive is 2008, where my dad is 39. Then my dad's age is always odd on February 29, so his age cannot possibly be twice my own.

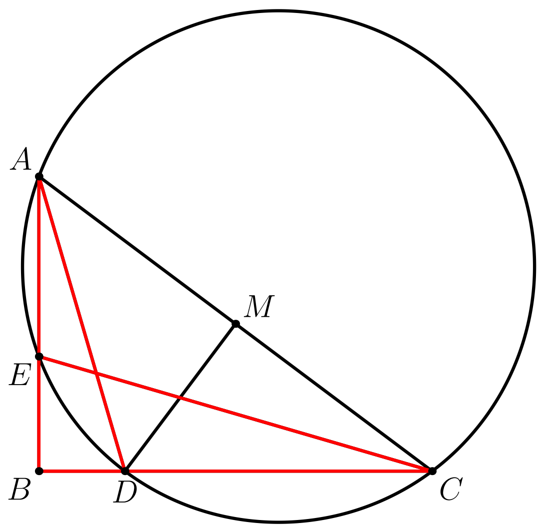
Therefore the answer is 365.

Problem 15. [**13**] Let $\triangle ABC$ be a triangle such that $AB = 3$, $BC = 4$, and $CA = 5$. Let D be the point on BC such that $AD = DC$, and let the circumcircle of $\triangle ADC$ intersect AB at $E \neq A$. Compute EC.

Proposed by Joseph Othman

Answer. $\boxed{\frac{25}{6}}$

Solution.



Let M be the midpoint of AC. Note that D is the intersection of the perpendicular bisector of line AC and segment BC. Then $DC = MC \cdot \frac{5}{4} = \frac{25}{8}$ by similar triangles $\triangle DMC$ and $\triangle ABC$. Now, since $\angle EAD = \angle ECD$, we have that $\triangle ABD \sim \triangle CBE$. Then $\frac{CE}{AD} = \frac{CB}{AB} = \frac{4}{3}$. Then $CE = \frac{4}{3} \cdot AD = \frac{4}{3} \cdot \frac{25}{8} = \frac{25}{6}$.

Problem 16. [**13**] Compute

$$(\log_2(20))(\log_2(40))(\log_2(640)) - (\log_2(10))(\log_2(160))(\log_2(320)).$$

Proposed by Rishabh Das

Answer. $\boxed{12}$

Solution. Let. $\log_2(10) = x$. Then this expression is

$$(x+1)(x+2)(x+6) - (x)(x+4)(x+5) = (x^3 + 9x^2 + 20x + 12) - (x^3 + 9x^2 + 20x) = 12.$$

Problem 17. [**14**] Adi, Das, and Big J are playing in a 3-person tournament of ping pong. Each of them is "equally good" at ping pong, so in any given game, the two players of that game are equally likely to win. After each game, the winner stays on the table, while the loser rotates off. The winner of the tournament is the first person to win 2 games in a row. Given that Adi and Das play the first match, what is the probability that Big J defies the odds to win the tournament?

Proposed by Joseph Othman

Answer. $\boxed{\frac{2}{7}}$

Solution. Let P be the probability that Big J wins the tournament given that he has won 0 game in a row at this point. Note that if Big J loses his first game, he does not win the tournament, since whoever won before him will have won two games in a row. Then there are 2 cases: Big J wins these two games to win the tournament, or he wins his first game, loses his second, and the tournament cycles back to him, each time with the winner of the previous game losing the next one until we return back to Big J playing with 0 games won in a row.

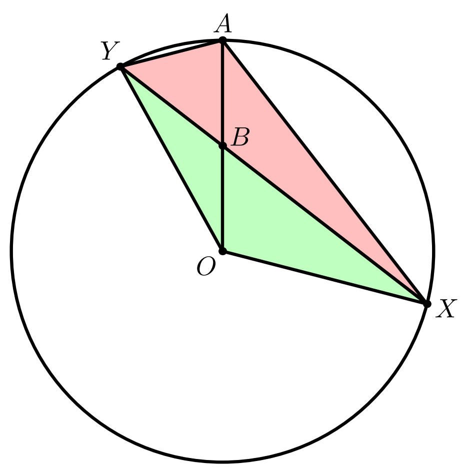
In case 1, Big J wins with probability $\frac{1}{2} \cdot \frac{1}{2} = \frac{1}{4}$. In case 2, Big J wins with probability $\frac{1}{2} \cdot \frac{1}{2} \cdot \frac{1}{2} \cdot P = \frac{P}{8}$, with each factor of $\frac{1}{2}$ corresponding to the probability that the winner of the previous game loses the current game. Then we have $P = \frac{1}{4} + \frac{P}{8}$. Solving this gives $P = \frac{2}{7}$ as the final answer.

Problem 18. [**14**] A circle Ω has center O and radius 8. Let A be a point on Ω, and B be the midpoint of OA. A line through B is drawn intersecting Ω at X and Y such that $\frac{BX}{BY} = 2$. Compute the area of $\triangle AXY$.

Proposed by Rishabh Das

Answer. $\boxed{6\sqrt{15}}$

Solution.



Let line AO hit Ω again at C. We have $AB = 4$ and $AC = 16$ so $BC = 12$. Calling $BY = m$ and applying Power of a Point gives $2m^2 = 48$, or $m = 2\sqrt{6}$, so $XY = 6\sqrt{6}$. Since B is the midpoint of AO, the distance from A to XY is equal to the distance from O to XY. Since $\triangle AXY$ and $\triangle OXY$ have the same base and same height, $[AXY] = [OXY]$. To compute $[OXY]$, note that $\triangle OXY$ is an isosceles triangle with $OX = OY = 8$ and $XY = 6\sqrt{6}$. Dropping a perpendicular from O to XY hitting XY at M gives $OM = \sqrt{10}$ by the Pythagorean Theorem. Then the final answer is $\frac{1}{2} \cdot 6\sqrt{6} \cdot \sqrt{10} = 6\sqrt{15}$.

Problem 19. [**15**] Suppose n is a randomly and uniformly selected 56-digit positive integer. Compute the probability that the sum of the digits of $n + 56$ is greater than the sum of the digits of n.

Proposed by Vidur Jasuja

Answer. $\boxed{\frac{31}{50}}$

Solution. Let $s(n) =$ the sum of the digits of n. The main claim is that $s(n+56) > s(n) \iff$ the addition of $n+56$ requires at most 1 carry. To see this, note that $s(n+56) = s(n)+s(56)-9c(n,56)$, where $c(x,y)$ represents the number of carries necessary to add x and y. This is true because initially the leftmost 2 digits increase by 5 and 6, respectively, and each carry that this sum induces will subtract 10 from the number in the digit where the carry occurs, and will add 1 to the digit to the left in the sum, subtracting 9 from the total sum of digits. Since we want $s(n+56) - s(n) > 0$ we need $s(56) - 9c(n,56) = 11 - 9c(n,56) > 0$, so $c(n,56) \leq 1$.

Now, we split into 2 cases.
Case 1: $c(n,56) = 0$
Then we require that the tens digit of n is less than 5, and the ones digit of n is less than 4. This occurs with probability $\frac{5}{10} \cdot \frac{4}{10} = \frac{1}{5}$.
Case 2: $c(n,56) = 1$
Note that the single carry must either occur in the tens place or the ones place. If it occurs in the ones place, then the ones digit of n must be at least 4 in order for this carry to occur, and the tens digit of n must be less than 4 in order to avoid a second carry. This occurs with probability $\frac{6}{10} \cdot \frac{4}{10} = \frac{6}{25}$.
If the carry occurs in the tens digit, then the ones digit must be less than 4, the tens digit must be at least 5, and the hundreds digit must be at most 8 in order to avoid a carry in the hundreds place. This occurs with probability $\frac{4}{10} \cdot \frac{5}{10} \cdot \frac{9}{10} = \frac{9}{50}$.
Then the total probability that $s(n+56) > s(n)$ is equal to $\frac{1}{5} + \frac{6}{25} + \frac{9}{50} = \frac{31}{50}$.

Problem 20. [Up to 28] Welcome to **USAYNO**!

Instructions: Submit a string of 6 *letters corresponding to each statement: put T if you think the statement is true, F if you think it is false, and X if you do not wish to answer. You will receive* $\frac{(n+1)(n+2)}{2}$ *points for n correct answers, but you will receive zero points if any of the questions you choose to answer are incorrect. Note that this means if you submit "XXXXXX" you will get one point.*

(1) Four positive integers form a nonconstant geometric sequence. Then their sum must have at least four positive integer factors.

(2) It is known that $1^3+2^3+3^3+\cdots+n^3 = \left(\frac{n(n+1)}{2}\right)^2$. There is some $k > 3$ for which $1^k+2^k+\cdots+n^k$ is equal to $f(n)^2$ for some polynomial f with rational coefficients.

(3) A triangle has integer side lengths and integer area. Its side lengths form an arithmetic sequence with common difference 4. Then its perimeter must be be a multiple of 8.

(4) All real polynomials P satisfying $P(x^2) = P(x)^2$ for all real x are either 0 or x^n for some nonnegative integer n.

(5) Let $\sigma(n)$ denote the sum of the divisors of n. Then

$$\sum_{k=1}^{n} \sigma(k) \leq n^2$$

for all positive integers n.

(6) Consider a triangle divided into 6 smaller triangles by 3 concurrent lines passing through its vertices. There exists such a configuration where the 6 triangles created have areas that form a non-constant geometric sequence in some order.

Proposed by Rishabh Das, Joseph Othman, and Josiah Moltz

Answer. $\boxed{\text{TFTTTF}}$

Solution. Only the second and sixth statements are false.

(1) Let the integers be xa^3, xa^2b, xab^2, xb^3, where one of a and b is nonzero. Then their sum is $x(a^3+a^2b+ab^2+b^3) = x(a+b)(a^2+b^2)$. Both of $a+b$ and a^2+b^2 are integers greater than 1, and are distinct, so this number has $1, a+b, a^2+b^2$, and $(a+b)(a^2+b^2)$ as factors, so it has at least four factors.

(2) Suppose such a k existed. Then $1 + 2^k = f(2)^2$, so $2^k = y^2 - 1$ for some integer y. However, this means that both $y - 1$ and $y + 1$ must be powers of 2, and the only y satisfying this is $y = 3$, and then $k = 3$ as well, a contradiction.

(3) Let the triangle have sides $k - 4, k$, and $k + 4$. Then its semi-perimeter is $\frac{3k}{2}$, so its area is

$$\sqrt{\left(\frac{3k}{2}\right)\left(\frac{k}{2}\right)\left(\frac{k}{2}+4\right)\left(\frac{k}{2}-4\right)} = \frac{k}{4}\sqrt{3(k^2-64)}$$

by Heron's formula. Immediately, we see k must be a multiple of 4, so let $k = 4m$. Then we require $3(16m^2 - 64)$ to be a perfect square, or $3m^2 - 12$ to be a perfect square. If m is odd, then $3m^2 - 12 \equiv 3 \pmod 4$, and is thus not a square. This means m is even, so $k = 8n$ for some n.

Then the perimeter of the triangle is $3k = 24n$, so it's a multiple of 8.

(4) Note that if r is a root, then so is r^2. Thus, if P has a complex root magnitude $d \neq 0, 1$, then it has a root with magnitude d^2, d^4, d^8, and so on, so it has infinitely many roots, so $P(x) \equiv 0$.

Otherwise, assume P has a root with magnitude 1, $\operatorname{cis}\theta$. If $\theta \neq 0°$, then P has roots $\operatorname{cis}\frac{\theta}{2^k}$ for all k, meaning it has infinitely many roots, a contradiction. If $\theta = 0°$, then note $P(-1)^2 = P(1) = 0$, so $P(-1) = 0$, so we can repeat the previous argument using $\theta = 180°$.

This means that P can only have 0 as a root, so it must be of the form x^n for some n.

(5) Note that

$$\sum_{k=1}^{n} \sigma(k) = \sum_{k=1}^{n}\sum_{m|k} m = \sum_{m=1}^{n} m \cdot \sum_{\substack{1\le k\le n \\ m|k}} 1 = \sum_{m=1}^{n} m \left\lfloor \frac{n}{m} \right\rfloor.$$

Each of the n terms of this summation is clearly at most n, so the entire sum is at most n^2.

(6) Suppose $\triangle ABC$ has three cevians, AD, BE, and CF, intersecting at a point P, and suppose

$$[BPD] : [CPD] : [CPE] : [APE] : [APF] : [BPF] = r^a : r^b : r^c : r^d : r^e : r^f,$$

where $\{a, b, c, d, e, f\} = \{0, 1, 2, 3, 4, 5\}$ and $r \neq 1$. Note that

$$\frac{BD}{DC} = \frac{[BPD]}{[CPD]} = r^{a-b},$$

and similarly $\frac{CE}{EA} = r^{c-d}$ and $\frac{AF}{FB} = r^{e-f}$. By Ceva's Theorem,

$$\frac{BD}{DC} \cdot \frac{CE}{EA} \cdot \frac{AF}{FB} = r^{a-b+c-d+e-f} = 1.$$

This means $a - b + c - d + e - f = 0$. However, $a - b + c - d + e - f \equiv a + b + c + d + e + f \equiv 15 \not\equiv 0 \pmod 2$, so this is a contradiction.

Problem 21. [16] Define $F_0 = 0, F_1 = 1$, and for $n \geq 1$, $F_{n+1} = F_n + F_{n-1}$. Compute the sum of the prime factors of

$$\sum_{n=1}^{10} F_{n+3}F_n.$$

Proposed by Rishabh Das

Answer. $\boxed{58}$

Solution. Note that

$$F_{n+3}F_n = (F_{n+2} + F_{n+1})(F_{n+2} - F_{n+1}) = F_{n+2}^2 - F_{n+1}^2.$$

Thus, the sum we have is

$$\sum_{n=1}^{10} = F_{n+3}F_n = \sum_{n=1}^{10} F_{n+2}^2 - F_{n+1}^2 = F_{12}^2 - F_1^2 = F_{12}^2 - 1,$$

where we use the fact that the sum telescopes. Since $F_{12} = 144$, we get $144^2 - 1 = 143 \cdot 145 = 11 \cdot 13 \cdot 5 \cdot 29$. The sum of these primes is 58.

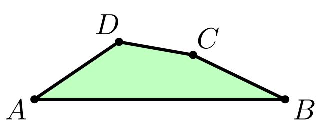
Problem 22. **[16]** In convex quadrilateral $ABCD$, $\angle A = 37°$, $\angle B = 23°$, $AB = 22$, $CD = 20$, and $AD = BC$. Compute $[ABCD]$, the area of quadrilateral $ABCD$.

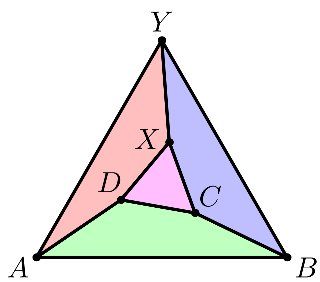
Proposed by Rishabh Das

Answer. $\boxed{7\sqrt{3}}$

Solution 1. The only relevant angle information is $\angle A + \angle B = 60°$.

Take three congruent copies of this quadrilateral, and arrange them putting the two sides of equal length next to each other, as shown.



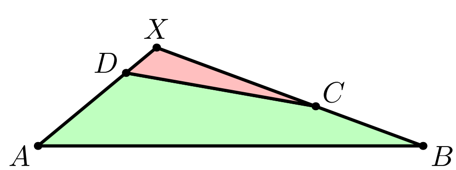


Also label the points as shown. We know from the angle condition that $\triangle ABY$ is equilateral, and since $CD = DX = XC$, $\triangle CDX$ is equilateral as well. The area of the region between these two equilateral triangles is

$$\frac{22^2\sqrt{3}}{4} - \frac{20^2\sqrt{3}}{4} = 21\sqrt{3}.$$

Since this region is the union of three congruent copies of the desired quadrilateral, the area of it is $7\sqrt{3}$.

Solution 2. Again the only angle information we need is $\angle A + \angle B = 60°$.



Extend AD and BC to intersect at X. Let $XD = x$, $XC = y$, and $AD = BC = a$. Because of the angle condition, $\angle AXB = 120°$.

The Law of Cosines gives

$$x^2 + xy + y^2 = 400$$

and

$$(x + a)^2 + (x + a)(y + a) + (y + a)^2 = 484 \implies (x^2 + xy + y^2) + 3(xa + ya + a^2) = 484.$$

Subtracting the two gives $xa + ya + a^2 = 28$. The area of the quadrilateral is

$$[ABX] - [CDX] = \frac{1}{2}\sin(120°)((x + a)(y + a) - xy) = \frac{\sqrt{3}}{4}(xa + ya + a^2) = 7\sqrt{3}.$$

Problem 23. **[17]** How many ordered pairs of positive integers (a, b) are there such that $a, b \leq 50$ and

$$\tau(ab) = \tau(a) + \tau(b) - 1,$$

where $\tau(n)$ is the number of divisors of n?

Proposed by Rishabh Das

Answer. $\boxed{152}$

Solution. Note that if either $a = 1$ or $b = 1$ then the equation is clearly satisfied. There are $50 + 50 - 1 = 99$ (where we subtract one for overcounting the case of $a = b = 1$) such cases. From here on out assume $2 \leq a, b \leq 50$.

We now appeal to a combinatorial argument. Let S_1 denote the set of divisors of a, and let S_2 denote the set of numbers that are a times a divisor of b. Note that the only element that S_1 and S_2 have in common are a, and S_1 and S_2 both consist only of divisors of ab. Thus,

$$\tau(ab) \geq |S_1 \cup S_2| = |S_1| + |S_2| - |S_1 \cap S_2| = \tau(a) + \tau(b) - 1.$$

Thus, we actually require equality to hold, i.e. every divisor of ab is either a multiple or a divisor of a. Similarly, every divisor of ab is either a multiple or a divisor of b.

Without loss of generality, assume $a \leq b$. Then we know b must be a multiple of a. If a prime p divides b but doesn't divide a, then p is neither a divisor nor a multiple of a, a contradiction. Suppose two primes p and q both divide b, with p also dividing a. Then $p^{\nu_p(ab)}$ isn't a divisor of b since it has more factors of p than b does, and isn't a multiple of b because it's not a multiple of q. Thus, this is a contradiction.

The only possibility left is if both a and b are powers of the same prime, p. We now perform cases on p.

If $p = 2$, then $a, b \in \{2, 4, 8, 16, 32\}$, giving 25 solutions.

If $p = 3$, then $a, b \in \{3, 9, 27\}$, giving 9 solutions.

If $p = 5$ or $p = 7$, then $a, b \in \{p, p^2\}$, giving 4 solutions.

Finally, for the eleven primes p satisfying $11 \leq p \leq 50$, then $a = b = p$, giving 1 solution.

Overall, there are $99 + 25 + 9 + 2 \cdot 4 + 11 \cdot 1 = 152$ ordered pairs.

Problem 24. **[17]** Let $f(n)$ denote the number of ordered pairs of positive integers (a, b) satisfying $ab = n$ and $a, b \leq 20$. Compute

$$\sum_{n=1}^{\infty} (-1)^n f(n) n.$$

Proposed by Rishabh Das and Joseph Othman

Answer. $\boxed{24100}$

Solution. Rewrite the desired sum as

$$\sum_{a=1}^{20} \sum_{b=1}^{20} (-1)^{ab} ab.$$

We will split this into two cases: ab being even and being odd. Then we want

$$\left(\sum_{a=1}^{20} \sum_{b=1}^{20} ab \right) - 2 \cdot \left(\sum_{\substack{1 \leq a, b \leq 20 \\ a, b \text{ are odd}}} ab \right).$$

The first summation is just $(1+2+3+\cdots+20)^2 = 210^2 = 44100$, while the other summation $(1+3+5+\cdots 19)^2 = 100^2 = 10000$. The final answer is then $44100 - 2 \cdot 10000 = 24100$.

Problem 25. [18] There is a unique quadratic function f such that

- there exists a constant c such that $f(f(x)) = c$ is satisfied only by $x \in \{1, 2, 3\}$, and
- $f(0) = 23$.

Compute c.

Proposed by Vidur Jasuja

Answer. $\boxed{53}$

Solution. Since $f(x)$ is symmetric about some vertical line (since it is quadratic), $f(f(x))$ must also be symmetric about the same axis[1], since if $f(a) = f(b)$, then $f(f(a)) = f(f(b))$.

Thus since the solutions of $f(f(x)) = c$ must also by symmetric about this axis, we realize that $x = 2$ must be the axis in question, since it is the only value for which the numbers $\{1, 2, 3\}$ are symmetric about.

If $f(x) = ax^2 + bx + c$ we know that the axis of symmetry is found at $x = -\frac{b}{2a}$, and so $-\frac{b}{2a} = 2$, meaning $b = -4a$. We also must have that $f(2)$ and $f(1)$ are symmetric about 2, since we have that $f(f(2)) = f(f(1))$, and we can't have $f(2) = f(1)$ (since if this were true f would be symmetric about $x = 1.5$).

Thus $f(2) + f(1) = 4$, and so $a(2)^2 - 4a(2) + c + a(1)^2 - 4a(1) + c = 4$. Remembering that $f(0) = 23$, we get $c = 23$, so we really have $4a - 8a + 23 + a - 4a + 23 = 4$.

Solving we get $-7a = -42$ so $a = 6$. Finally $f(x) = 6x^2 - 24x + 23$, and so

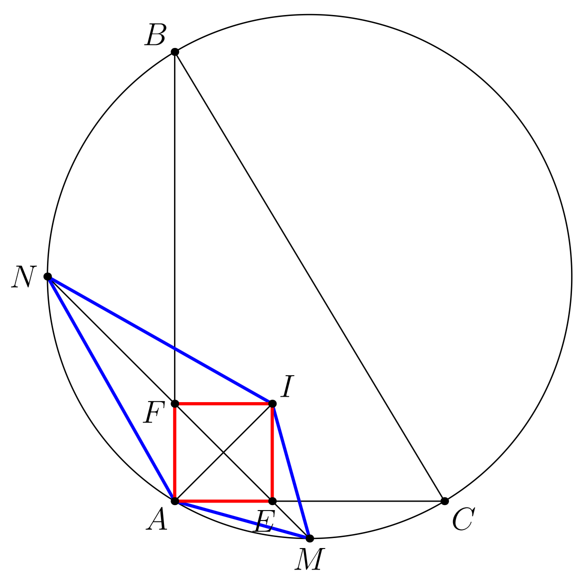
$$c = f(f(2)) = f(6(2)^2 - 24(2) + 23) = f(-1) = 6(-1)^2 - 24((-1)) + 23 = 53$$

Problem 26. [18] Let the incircle of triangle ABC touch sides AC and AB at E and F, respectively. If $EF = 42$ and line EF bisects arcs $\widehat{AC}$ and $\widehat{AB}$ of the circumcircle of triangle ABC, find the area of quadrilateral $AEIF$.

Proposed by Josiah Moltz

Answer. $\boxed{882}$

Solution.



[1] Another justification for this is that if $f(x) = a(x - r_1)(x - r_2)$ (where the roots may be complex), then $f(f(x)) = a(f(x) - r_1)(f(x) - r_2)$, and since $f(x) - r_1$ and $f(x) - r_2$ maintain the same b terms as $f(x)$, both are still symmetric about the same axis. Thus since $f(f(x))$ is the product of two functions symmetric about the same axis as $f(x)$, $f(f(x))$ is symmetric about the same axis as f.

By the incenter-excenter lemma, we know that $AN = IN$, and $AM = IM$. Thus, $AMIN$ is a kite, and MN is the perpendicular bisector of AI, which means EF is also a perpendicular bisector of AI.

This tells us that we have $AF = FI$ and $IE = EA$. However we already had $IF = IE$, and $\angle AFI = \angle IEA = 90°$ from the fact that I was an incenter. This is enough to tell us that $AEIF$ is a square.

We are given the length of a diagonal of this square is 42, so the area of the square is $\frac{42^2}{2} = 882$.

Problem 27. **[19]** Suppose that $P(x)$ is a polynomial of degree at most 5 such that $P(2^k) = k$ for $k = 1, 2, 3, 4, 5$, and 6. Compute $P(2^7)$.

Proposed by Rishabh Das and Joseph Othman

Answer. $\boxed{9772}$

Solution. Let $Q(x) = P(2x) - P(x) - 1$. Note that Q is also a polynomial with degree at most 5, and $Q(2) = Q(4) = Q(8) = Q(16) = Q(32) = 0$. This means that these five values must be the five roots of Q, so

$$Q(x) = c(x-2)(x-4)(x-8)(x-16)(x-32)$$

for some constant c. However, plugging in $x = 0$ gives $Q(0) = P(0) - P(0) - 1 = -1$, so

$$-1 = c(-2)(-4)(-8)(-16)(-32) = -c \cdot 2^{15},$$

meaning $c = \frac{1}{2^{15}}$. Now plugging in $x = 64$, we have $Q(64) = P(128) - P(64) - 1 = P(128) - 7$, so

$$P(128) - 7 = \frac{1}{2^{15}}(64-2)(64-4)(64-8)(64-16)(64-32) = (32-1)(16-1)(8-1)(4-1)(2-1) = 9765.$$

Then $P(128) = 9772$.

Problem 28. **[19]** In a square grid with 11 rows and 10 columns, every square not in the top row is randomly assigned a right arrow or an up arrow. You can only move from one square to the adjacent square where the arrow is pointing. What is the probability that there is a path from some square in the bottom row to some square in the top row?

Proposed by Paul Gutkovich

Answer. $\boxed{\frac{1}{2}}$

Solution. Consider adding a column to the right of this grid (without any arrows, like the top row). We claim that an equivalent condition is that the path from the bottom left corner ends in the top row. Clearly this is sufficient; to prove that it is necessary, assume for the sake of contradiction that there does not exist a path from the bottom right corner of the grid to the top row (i.e. the path from this corner ends in the right column of the grid, and not in the top right corner). Furthermore, assume that there is a path from any other bottom tile to the top row.

We claim that this is impossible. Consider drawing a line segment between each two consecutive squares in the path from the bottom left corner. Do the same for each pair of consecutive squares in the path from the bottom tile that ends in the top row. These paths must intersect at some square, since one path starts on the left and ends on the right of the other. But this cannot happen because if the paths intersect, they must end at the same spot (since each move is determined by the arrow in the square, which is constant). This is a contradiction, since one path ends at the top row and the other doesn't.

Now, we reduce to solving the new problem: what is the probability that the path from the bottom left square ends in the top row?

We solve this by appealing to a symmetry argument. Say that in some grid, the path from the bottom left corner ends in the top row. Then consider each square that this path visited reflected over the line $y = x$. We can create a new grid that uses each of these squares and ends in the right column, by taking the arrow in the preimage of each square, swapping it, and putting it in the new square. This reflects the path over the line $y = x$. Since this map is bijective, exactly half of the grids satisfy that the path from the bottom left square ends in the top row, making the probability $\frac{1}{2}$.

Problem 29. **[20]** Define

$$f(n) = \sum_{wxyz|n} \frac{1}{wxyz},$$

where the summation runs over all ordered quadruples of positive integers (w, x, y, z) satisfying $wxyz \mid n$. Let the set A consist of numbers of the form $2^p 3^q 7^r$, where p, q, and r are nonnegative integers.

Find the smallest value of M satisfying $f(a) \le M$ for all $a \in A$.

Proposed by Josiah Moltz

Answer. $\boxed{\dfrac{2401}{16}}$

Solution. The reason four variables is annoying is that we have lots of repeats, since $wxyz = 2$ is generated by 4 different quadruples of (w, x, y, z).

Repeats of $wxyz$ actually occur when a fixed number of 2s, 3s, and 7s are split among the four variables in different ways. We can condense this to be $\frac{\text{number of repeats}}{wxyz}$. In order to count repeats we will count the number of ways to distribute the 2s 3s and 7s among w, x, y, z

If we have $\hat{p}$ 2s, and want to split them up among w, x, y, and z, we can use stars and bars to see that there are $\binom{\hat{p}+3}{3}$ ways to do this.

Thus, a factor of $\frac{1}{2^{\hat{p}}}$ is actually contributed as $\frac{\binom{\hat{p}+3}{3}}{2^{\hat{p}}}$.

This idea can be repeated for 3 and 7 as well.

Since we no longer have to think about repeats, we can now focus on building the unique values of $wxyz$, which are actually just the unique factors of a.

Thus, if $a = 2^p 3^q 7^r$, then all factors of a are composed of a unique power of 2, 3, and 7. Thus

$$f(a) = \left(\sum_{i=0}^{p} \frac{\binom{i+3}{3}}{2^i}\right) \left(\sum_{j=0}^{q} \frac{\binom{j+3}{3}}{3^j}\right) \left(\sum_{k=0}^{r} \frac{\binom{k+3}{3}}{7^k}\right)$$

The larger p, q, and r become, the larger each of our sums will actually get, increasing the value of $f(a)$. This means that as p, q, and r get infinitely large, f approaches

$$\left(\sum_{i=0}^{\infty} \frac{\binom{i+3}{3}}{2^i}\right) \left(\sum_{j=0}^{\infty} \frac{\binom{j+3}{3}}{3^j}\right) \left(\sum_{k=0}^{\infty} \frac{\binom{k+3}{3}}{7^k}\right).$$

Noting that

$$\sum_{i=0}^{\infty} \binom{i+3}{3} x^i = \left(\frac{1}{1-x}\right)^4,$$

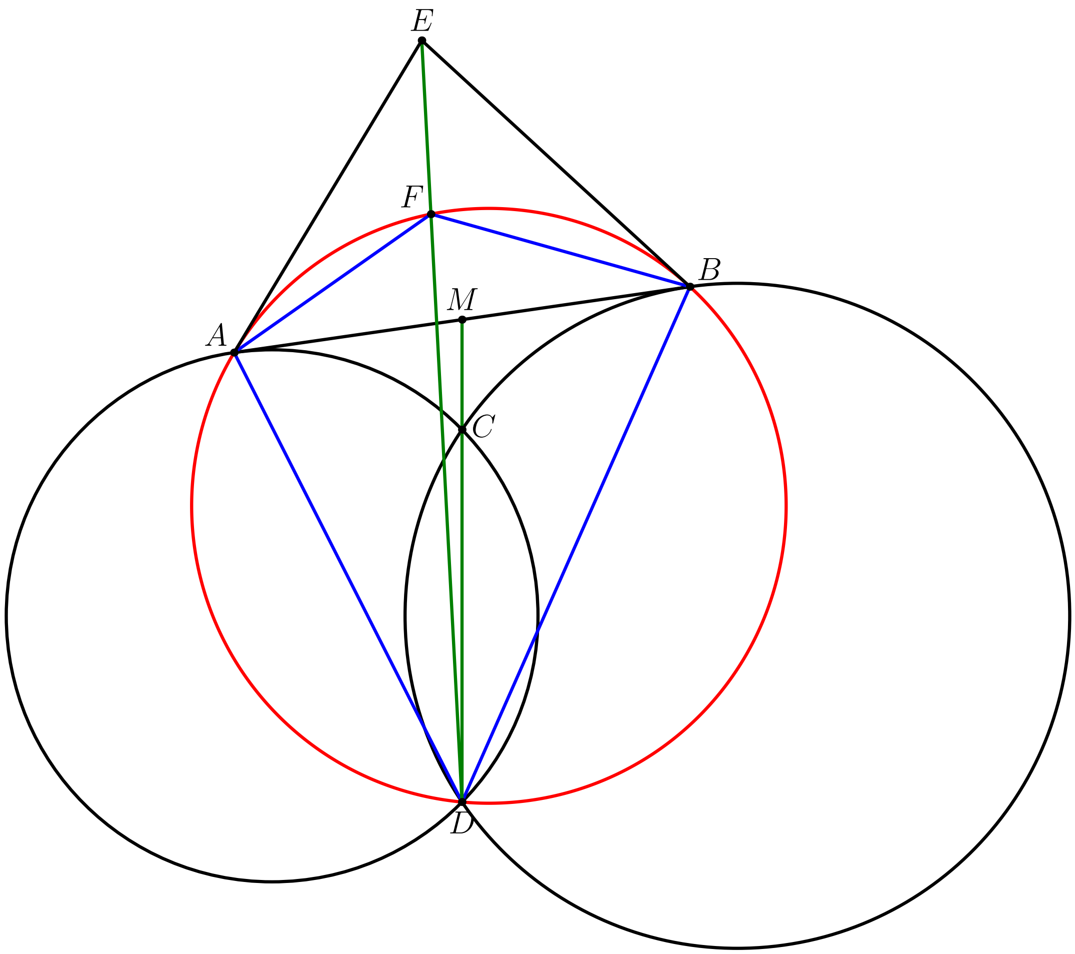
we get that the smallest upper bound on f is $\left(\frac{1}{1-\frac{1}{2}}\right)^4 \left(\frac{1}{1-\frac{1}{3}}\right)^4 \left(\frac{1}{1-\frac{1}{7}}\right)^4$, which is $\frac{2401}{16}$.

Problem 30. **[20]** Let ω and Γ be circles with radii 15 and 25, respectively, such that the centers of ω and Γ are 30 apart. Say that ω and Γ have common external tangent AB with A on ω, B on Γ, and say that ω and Γ intersect at C and D with C closer to AB than D. Let F be the reflection of C across AB. Compute DF.

Proposed by Joseph Othman

Answer. $\boxed{\dfrac{20\sqrt{30}}{3}}$

Solution 1.



Firstly, we claim that points A, F, D and B are cyclic. To prove this, note that $\angle AFB = \angle ACB$ by reflection, so we must show that $\angle ACB + \angle ADB = 180^\circ$. But

$$\angle ACB = 180^\circ - \angle BAC - \angle ABC = 180 - \angle ADC - \angle BDC = 180^\circ - \angle ADB$$

by the tangency condition, so we have proven our claim.

Let O be the center of $(AFBD)$ let O_1 and O_2 be the centers of the circles of radii 15 and 25, respectively, and let E be the intersection of the tangents of A and B to $(AFBD)$. Finally, let M be the intersection of lines AB and CD. Note that by Power of a Point,

$$AM^2 = MC \cdot MD = MB^2,$$

so M is the midpoint of AB.

We now claim that points E, F, and C are collinear. Note that since ED is the D-symmedian of $\triangle ABD$, it suffices to show that $\angle ADF = \angle BDM$. Using cyclic quadrilateral $AFBD$, the tangency of AB, and angle properties of reflection, we can angle chase to get

$$\angle ADF = \angle ABF = \angle ABC = \angle BDC = \angle BDM,$$

and so we have proven our collinearity.

Now, consider force-overlaid inversion centered at D with radius $\sqrt{AD \cdot BD}$. Our main claim is that F and M are images of each other, as are E and C. First, focus on point M. It is the intersection of lines DC and AB. Then under the inversion, since DC is a median, its image is the D-symmedian of $\triangle ADB$, and the points A and B swap, so the image of AB is (ABD). Then the image of M is the intersection of these two objects, which is either F or D; clearly its image cannot be D, so M must map to F. Then F maps to M since inversions map points to each other.

As for point E, note that its image must lie on line MD. Let its image be E'. Then we know that $\angle EAD = \angle BE'D$. Exactly one point on MD satisfies this; we claim that C is this unique point. Note that

$$\begin{aligned}\angle EAD &= \angle EAF + \angle DAF \\ &= 180^\circ - \angle AFD \\ &= 180^\circ - \angle ABD \\ &= 180^\circ - \angle CBD - \angle CBA \\ &= 180^\circ - \angle CBD - \angle CDB = \angle BCD\end{aligned}$$

so $\angle EAD = \angle BCD$ and so E and C are images of each other as well.

Now note that $ED \cdot CD = AD \cdot BD = FD \cdot MD$, so $\frac{FD}{CD} = \frac{ED}{MD}$, so $EM \parallel FC$. Then if we are able to calculate $\frac{ED}{MD}$ and CD, then we can compute FD and finish. Since $EM \parallel FC$, we also have that $\frac{ED}{MD} = \frac{EF}{MC}$. Let this common ratio be X. Then

$$X^2 = \frac{ED}{MD} \cdot \frac{EF}{MC} = \frac{\text{pow}(E, (AFBD))}{\text{pow}(M, (AFBD))} = \frac{EA^2}{MA^2},$$

So $X = \frac{EA}{MA}$. Then we have that $\frac{FD}{CD} = \frac{EA}{MA}$. We can compute $MA = \frac{AB}{2} = 10\sqrt{2}$ and $CD = \frac{20\sqrt{14}}{3}$ by taking twice the height from C to the line O_1O_2. Then it remains to compute EA.

To do this, we will apply the Law of Cosines on $\triangle EAB$. Since $\angle EAO = \angle EBO = 90^\circ$, A, B, O and E are cyclic, so $\angle AEB = 180^\circ - \angle AOB$. We claim that $\angle AOB = \angle O_1DO_2$. Angle chasing gives

$$\begin{aligned}\angle AOB &= 180^\circ - \angle OAB - \angle OBA \\ &= 90^\circ - \angle OAB + 90^\circ - \angle OBA \\ &= \angle O_1AO + \angle O_2BO \\ &= \angle O_1DO + \angle O_2DO = \angle O_1DO_2.\end{aligned}$$

We can compute $\cos(\angle O_1DO_2) = -\frac{1}{15}$ so $\cos(\angle AEB) = \frac{1}{15}$. Then the Law of Cosines on $\triangle EAB$ gives

$$EA^2 + EA^2 - 2 \cdot EA^2 \cdot \frac{1}{15} = AB^2 = 800$$

which yields $EA = 10\sqrt{\frac{30}{7}}$. To finish, we use our identity that $\frac{FD}{CD} = \frac{EA}{MA}$ with all of the lengths that we know:

$$\begin{aligned}\frac{FD}{CD} = \frac{EA}{MA} &\implies \frac{FD}{\frac{20\sqrt{14}}{3}} = \frac{10\sqrt{\frac{30}{7}}}{10\sqrt{2}} \\ &\implies FD = \frac{20\sqrt{14}}{3} \times \sqrt{\frac{15}{7}} \\ &\implies FD = \frac{20\sqrt{30}}{3}.\end{aligned}$$

Solution 2. We pick up from the point where we realize F and M are images of each other under a force-overlaid inversion centered at D with radius $\sqrt{AD \cdot BD}$

Since M and F our inverses, $DF = \frac{AD \cdot BD}{DM}$. We can use power of a point to compute DM, but $AD \cdot BD$ is a bit harder to compute.

Looking at triangle ABD, law of sines gives us $\frac{BD}{AD} = \frac{\sin\angle BAD}{\sin\angle DBA}$

However looking at triangles AO_1D, $AD = 2AO_1 \sin\frac{1}{2}\angle AO_1D$, but since AB is tangent to ω, we know that $\frac{1}{2}\angle AO_1D = \angle BAD$. Thus $AD = 2AO_1 \sin\angle BAD = 30\sin\angle BAD$.

Symmetrically, $BD = 50\sin\angle DBA$. Thus $\frac{50\sin\angle DBA}{30\sin\angle BAD} = \frac{\sin\angle BAD}{\sin\angle DBA}$, meaning $\frac{BD}{AD} = \frac{\sin\angle BAD}{\sin\angle DBA} = \sqrt{\frac{50}{30}} = \sqrt{\frac{5}{3}}$.

Now set $AD = \sqrt{3}y$ and $BD = \sqrt{5}y$. We want to find $\sqrt{15}y^2$, and know that we need to compute DM. To involve a y^2 and a DM, we can use Stewarts on triangle ABD to get

$$AD^2\frac{AB}{2} + BD^2\frac{AB}{2} = \left(\frac{AB}{2}\right)^2 AB + (DM)^2 AB$$

$$AD^2 + BD^2 = \frac{AB^2}{2} + 2DM^2$$

Now using our values involving y we get

$$8y^2 = \frac{AB^2}{2} + 2DM^2$$

$$\frac{\sqrt{15}y^2}{DM} = \sqrt{15}\frac{\frac{AB^2}{4} + DM^2}{4DM}$$

Now we can compute DM by using power of a point on point M, since $AM^2 = MC(MD)$ so $\frac{AB^2}{4} = (DM - CD)DM$. Substituting this in, we get

$$\frac{\sqrt{15}y^2}{DM} = \sqrt{15}\frac{2DM - CD}{4}$$

Now since $AB = 20\sqrt{2}$ and $CD = \frac{20\sqrt{14}}{3}$ (from the first solution), we have

$$200 = DM^2 - \frac{20\sqrt{14}}{3}DM$$

We can now solve for DM with the quadratic formula and reject the negative value getting

$$DM = \frac{\frac{20\sqrt{14}}{3} + \sqrt{\frac{5600}{9} + 800}}{2} = \frac{10\sqrt{14}}{3} + \frac{40\sqrt{2}}{3}$$

So our final answer is

$$\sqrt{15}\frac{\frac{20\sqrt{14}}{3} + \frac{80\sqrt{2}}{3} - \frac{20\sqrt{14}}{3}}{4} = \frac{20\sqrt{30}}{3}$$