Stuyvesant Team Contest: Solutions

Fall 2021

Problem 1. **[6]** Compute

$$\left(\frac{1}{1+2+4+8+16+32}\right)\cdot\left(\frac{1}{1}+\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\frac{1}{16}+\frac{1}{32}\right).$$

Proposed by Rishabh Das

Answer. $\boxed{\frac{1}{32}}$

Solution. The value is

$$\left(\frac{1}{1+2+4+8+16+32}\right)\cdot\left(\frac{32+16+8+4+2+1}{32}\right)=\frac{1}{32}.$$

Note that we didn't actually need to compute $1+2+4+8+16+32$.

Problem 2. **[6]**

We're no strangers to love
You know the rules and so do I
A full commitment's what I'm thinking of
You wouldn't get this from any other guy

I just wanna tell you how I'm feeling
Gotta make you understand

Never gonna give you up
Never gonna let you down
Never gonna run around and desert you
Never gonna make you cry
Never gonna say goodbye
Never gonna tell a lie and hurt you

Proposed by Rishabh Das, Vidur Jasuja, and Jerry Liang

Answer. $\boxed{32}$

Solution. Problem 3 says the product of the answers to problems 1, 2, and 3 is equal to the answer to problem 3. This means that either the answer to problem 3 is 0, or the product of the answers to problems 1 and 2 is 1. Evidently the answer to problem 3 isn't 0 (we could also check this by noting that this problem isn't solvable if it were), so the answer to this problem is the reciprocal of the answer to problem 1, or 32.

Problem 3. [7] The answer to this problem is equal to the product of the answers to problems 1, 2, and 3.

Suppose problem k on NYCTC is worth $\lfloor \frac{11+k}{2} \rfloor$ points, except for problem 20, which is worth 28 points. Find the sum of all n such that problem n is worth n points.

(Here, $\lfloor x \rfloor$ is equal to the largest integer that is at most x. For example, $\lfloor \pi \rfloor = 3$ and $\lfloor 13 \rfloor = 13$.)

Proposed by Rishabh Das

Answer. $\boxed{21}$

Solution. Problem n is worth either $\frac{11+n}{2}$ points are $\frac{11+n}{2} - \frac{1}{2} = \frac{10+n}{2}$ points. Setting n equal to $\frac{11+n}{2}$ gives $n = 11$, and setting n equal to $\frac{10+n}{2}$ gives $n = 10$. The sum of these two values is 21.

Problem 4. [**7**] The number $\underline{A}\,\underline{0}\,\underline{2}\,\underline{0}\,\underline{2}\,\underline{1}$ is divisible by 7, where A is a digit. What is A?

Proposed by Rishabh Das

Answer. $\boxed{6}$

Solution. The number $\underline{A}\,\underline{0}\,\underline{2}\,\underline{0}\,\underline{2}\,\underline{1}$ is divisible by 7 if and only if the number $\underline{A}\,\underline{0}\,\underline{2}$ is divisible by 7, since all we're doing is multiplying this by 1000 and adding 21.

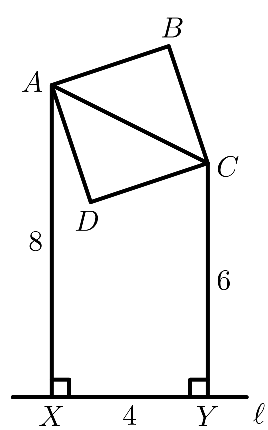
From here we could just test values of A to get that 6 works. Alternatively, adding 28 to this number gives $\underline{A}\,\underline{3}\,\underline{0}$ is divisible by 7, so $\underline{A}\,\underline{3}$ is divisible by 7, from which it's easy to see that $A = 6$.

Problem 5. [**8**] Square $ABCD$ and line ℓ lie on a plane. Let X and Y be the feet of the perpendiculars from A and C to line ℓ, respectively. If $AX = 8, CY = 6$, and $XY = 4$, compute the area of $ABCD$.

Proposed by Rishabh Das

Answer. $\boxed{10}$

Solution.



If we orient ℓ to be parallel to the x-axis, the x-displacement from A to C is 4, while the y-displacement is $8-6 = 2$. This means the length of AC is $\sqrt{4^2 + 2^2} = \sqrt{20}$, so the side length of square $ABCD$ is $\frac{\sqrt{20}}{\sqrt{2}} = \sqrt{10}$. Thus, the area of square $ABCD$ is 10.

If ℓ passes through the square, it's possible the area of the square is 106. Both 10 and 106 were accepted during the contest.

Problem 6. [**8**] A fair coin is flipped 7 times. What is the probability the product of the number of heads and the number of tails is a multiple of 3?

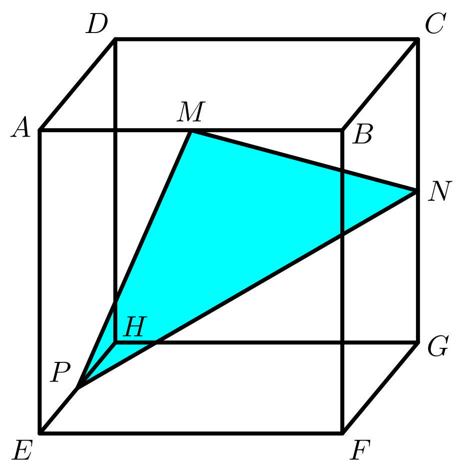
Proposed by Rishabh Das

Answer. $\boxed{\frac{43}{64}}$

Solution. The only way this product is not a multiple of 3 is if both the number of heads and number of tails are not multiples of 3. The only way this can happen is if we flip 2 of one of them and 5 of the other.

There is a $\frac{\binom{7}{2}}{2^7} = \frac{21}{128}$ probability we flip 2 heads and 5 tails, and similarly a $\frac{21}{128}$ probability we flip 2 tails and 5 heads. Overall, there's a $\frac{21}{64}$ probability that the product of the number of heads and number of tails is not a multiple of three. This makes the answer $1 - \frac{21}{64} = \frac{43}{64}$.

Problem 7. [**9**] Suppose $ABCDEFGH$ is a cube of side length 4, as shown. Suppose M, N, and P are the midpoints of AB, CG, and HE, respectively. What is the area of $\triangle MNP$?



Proposed by Rishabh Das

Answer. $\boxed{6\sqrt{3}}$

Solution. We can find

$$PM = \sqrt{PE^2 + EA^2 + AM^2} = \sqrt{2^2 + 4^2 + 2^2} = \sqrt{24}.$$

Similarly, $MN = NP = \sqrt{24}$. Thus, $\triangle MNP$ is equilateral. The area of the triangle is then

$$\frac{(\sqrt{24})^2\sqrt{3}}{4} = 6\sqrt{3}.$$

Remark. We can also note $\triangle MNP$ is equilateral since $\triangle ACH$ and $\triangle BGE$ are both equilateral, and then use mean geometry to get the midpoints of AB, CG, and EH form an equilateral triangle.

Problem 8. [**9**] There are 10 lily pads arranged in a circle. Kelvin the frog is currently on a lily pad, and every minute either jumps 2 lily pads clockwise or 3 lily pads counterclockwise with equal probability. After 5 minutes, what is the probability Kelvin is back where he started?

Proposed by Rishabh Das

Answer. $\boxed{\frac{1}{2}}$

Solution. Modify the given process so that Kelvin always jumps 2 lily pads clockwise, and then either stays in places or jumps to the lily pad diametrically opposite from him, with equal probability. After 5 minutes, Kelvin has jumped two lily pads clockwise 5 times, which is equivalent to doing nothing. Thus, Kelvin lands back where he started if and only if he jumps diametrically opposite an even number of times. The probability Kelvin does this is $\frac{1}{2}$, since Kelvin's last move is the only thing that determines if he ends up where he started or not.

Problem 9. **[10]** Suppose $225 \cdot 226 + 1 = pq$ for primes $p > q$. Compute $2p + q$.

Proposed by Rishabh Das

Answer. $\boxed{693}$

Solution 1. Let $x = 15$. Then our number is

$$x^2(x^2+1)+1 = x^4 + x^2 + 1 = (x^2+1)^2 - x^2 = (x^2+x+1)(x^2-x+1).$$

Thus, $225. \cdot 226 + 1 = 241 \cdot 211$. We may check that these are prime, but these are also given to us by the problem statement. Thus, $p = 241$ and $q = 211$. Then $2p + q = 482 + 211 = 693$.

Solution 2. We may write

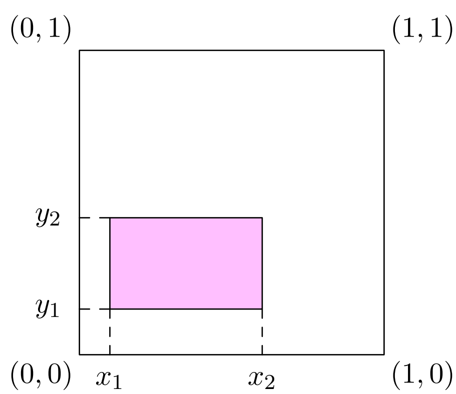
$$225 \cdot 226 + 1 = 226(226-1) + 1 = 226^2 - 225 = 226^2 - 15^2 = (226+15)(226-15) = 241 \cdot 211.$$

Then we can proceed as in solution 1.

Problem 10. **[Up to 10]** Pick a rectangle fully contained within the unit square $\mathcal{S} = [0,1] \times [0,1]$ and with sides parallel to $\mathcal{S}$.

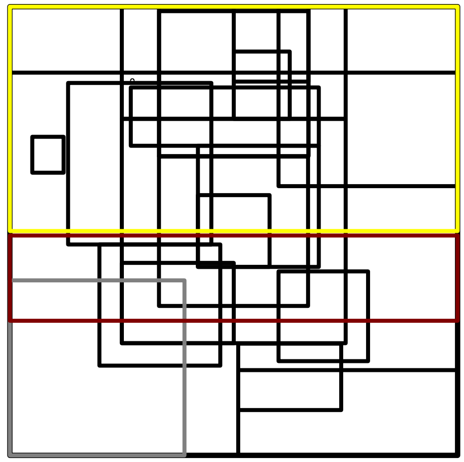
Let T be the number of submissions. Let I be the number of rectangles other teams submit that intersect with yours (sharing an edge or vertex counts as intersection). Let A be the area of your rectangle. Your score is $\lfloor cA(T-I-1) \rfloor$, where $c = \frac{10}{\max{(T-I-1)A}}$ over all submissions.

Your answer should be submitted in the form (x_1, x_2, y_1, y_2), where your rectangle is $[x_1, x_2] \times [y_1, y_2]$ (in particular, your rectangle will have vertices $(x_1, y_1), (x_1, y_2), (x_2, y_1)$, and (x_2, y_2)). Your submission should satisfy $0 \le x_1 < x_2 \le 1$ and $0 \le y_1 < y_2 \le 1$. Invalid submissions will result in 0 points.



Proposed by Vidur Jasuja

Answer. $\boxed{\text{N.A.}}$ Below are the results, where the first, second, and third best teams are marked in gold, silver, and bronze, respectively.



Problem 11. [11] A group of 103 people, including Taylor and Kanye, will form a 5-person team. The captain of the team (who is a member of the team) is either Taylor or Kanye. If Taylor is the captain, Kanye refuses to also be on the team. However, if Kanye is the captain of the team, then Taylor is okay with being on the team. If a team is randomly selected from all possible teams, compute the probability Kanye is on the team.

Proposed by Rishabh Das and Vidur Jasuja

Answer. $\boxed{\frac{51}{100}}$

Solution. Kanye can only be on the team if he is the captain, since otherwise Taylor is the captain, and he will refuse to be on the team.

If Taylor is the captain of the team, there are $\binom{101}{4}$ ways to choose the rest of the team, as we need to choose four people from the 101 people that are not Taylor or Kanye.

If Kanye is the captain of the team, there are $\binom{102}{4}$ ways to choose the rest of the team, as we need to choose four people from the 102 people that are not Kanye.

The probability Kanye is on the team is

$$\frac{\binom{102}{4}}{\binom{102}{4}+\binom{101}{4}} = \frac{102 \cdot 101 \cdot 100 \cdot 99}{101 \cdot 100 \cdot 99 \cdot (102+98)} = \frac{102}{200} = \frac{51}{100}.$$

Problem 12. [11] There is a unique triple of primes (p, q, r) satisfying $2pqr - 5p^2 - 5q^2 + 5r^2 = 0$ and $p < q$. Find (p, q, r).

Proposed by Vidur Jasuja

Answer. $\boxed{(2, 7, 5)}$

Solution. One of p, q, r must be 5, so that the expression is a multiple of 5. Furthermore, if all of p, q, r are odd, then the expression is odd, so one of p, q, r is equal to 2.

If $r = 5$, then we have

$$10pq - 5p^2 - 5q^2 + 125 = 0 \implies p^2 - 2pq + q^2 = 25 \implies (p-q)^2 = 5.$$

Then, evidently, $p = 2$ and $q = 7$, yielding the triple $(2, 7, 5)$, so we're done.

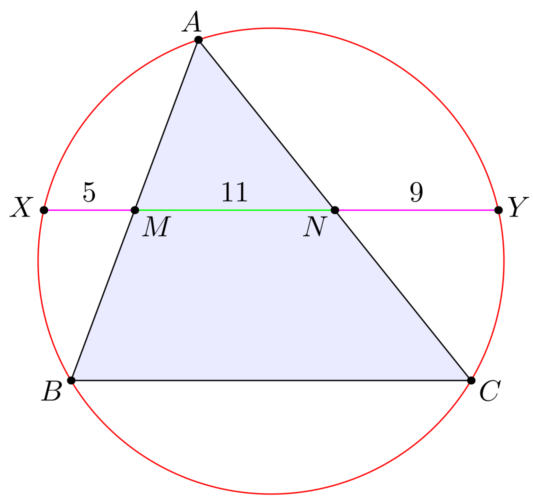
As a further note, simple casework on the case $p = 5$ (and, analogously, $q = 5$), will yield no solutions.

Problem 13. [**12**] Suppose M and N are the midpoints of sides AB and AC of $\triangle ABC$. Let line MN intersect the circumcircle of $\triangle ABC$ at X and Y such that M is between X and N. If $XM = 5, MN = 11$, and $NY = 9$, compute the area of $\triangle ABC$.

Proposed by Rishabh Das

Answer. $\boxed{33\sqrt{39}}$

Solution.



By power of a point, $XM \cdot MY = AM \cdot BM = AM^2$, so $AM^2 = 5 \cdot (11 + 9) = 100$, so $AM = 10$. This means $AB = 20$.

Again by power of a point, $YN \cdot NX = AN \cdot BN = AN^2$, so $AN^2 = 9 \cdot (11 + 5) = 144$, so $AN = 12$. This means $AC = 24$.

Since MN is the A-midline of $\triangle ABC$, $BC = 2 \cdot MN = 22$.

Thus, $\triangle ABC$ has side lengths $20, 22$, and 24. The area, by Heron's formula, is

$$\sqrt{33 \cdot 13 \cdot 11 \cdot 9} = 33\sqrt{39}.$$

Problem 14. [**12**] How many ways can the cells of a 3×3 grid be colored blue and red such that no row or column has all three of its cells of the same color?

Proposed by Rishabh Das

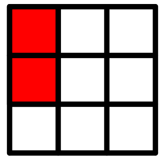
Answer. $\boxed{102}$

Solution. Assume that the majority of the cells are red, and multiply the total we get at the end by 2.

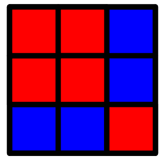
If more than 6 of the cells are red, then one row will be fully red. Thus, either 5 or 6 of the cells will be red.

First assume 6 of the cells are red. Then there are only 3 blue cells. The blue cells must all be on different rows and columns in order for no row or column to be monochromatic. There are 3 ways to choose where the blue cell goes in the first row, 2 remaining ways to choose where it goes in the second row, and 1 remaining possibility for the last row. Thus, there are $3 \cdot 2 \cdot 1 = 6$ possibilities in this case.

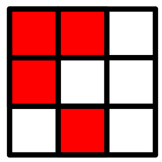
Now assume 5 of the cells are red. We require two columns to have 2 red cells and the last one to have 1 red cell. There are 3 ways to choose which columns have 2 red cells. Without loss of generality, assume the first two columns have 2 red cells. There are 3 ways to choose which red cells in the first column are red; assume the below configuration.



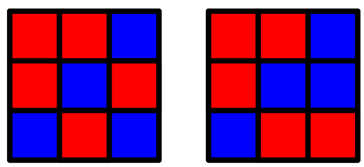
There are three cases for the second column. If the two cells are the ones in the first two rows, then the last red cell is in the bottom right.



The other two cases are symmetric; assume the one below.



Then there are two cases for the last red cell, as the only condition is that it can't be in the first row.



Thus, this case gives $3 \cdot 3 \cdot (1 + 2 \cdot 2) = 45$ possibilities.

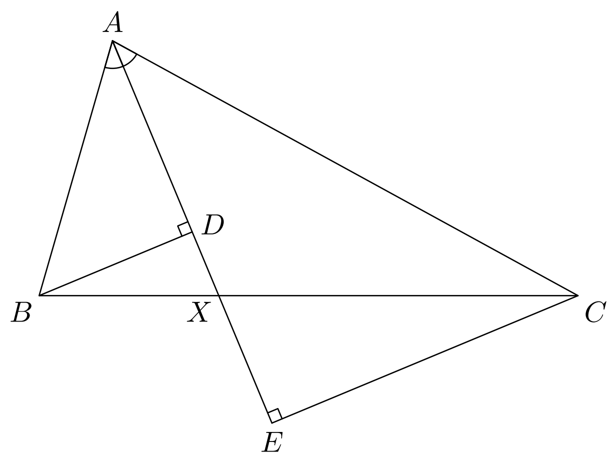
Overall, there are $6 + 45 = 51$ ways for no row or column to be monochromatic where the majority of cells are red, so there are $2 \cdot 51 = 102$ ways with no restriction.

Problem 15. **[13]** Let ℓ be the A-angle bisector of triangle ABC. Let the feet from B and C to ℓ be D and E, respectively. If $BD = 12, CE = 24$, and $DE = 15$, find the area of triangle ABC.

Proposed by Jerry Liang

Answer. $\boxed{360}$

Solution. Let X be the intersection of ℓ with BC.



Since $BD \perp AD$ and $CE \perp AE$, we have that $BD \parallel EC$. Therefore, $\triangle BDX \sim \triangle CEX$. Since $\frac{BD}{CE} = \frac{12}{24} = \frac{1}{2}$, we have $\frac{DX}{EX} = \frac{1}{2} \implies DX = \frac{DE}{3} = 5$ and $EX = \frac{2DE}{3} = 10$.

By the Pythagorean theorem, $BX = 13$ and $XC = 26$.

Since $\angle BAD = \angle DAC$, by AA similarity we have that $\triangle BAD \sim \triangle CAE$ with ratio $BD : CE = 1 : 2$. Therefore, $AD = \frac{1}{2}AE \implies AD = DE = 15$.

Now we can compute the area of triangle ABC as follows:

$$[ABC] = [ABX] + [ACX] = [ABD] + [BDX] + [ACE] - [CEX]$$

$$= \frac{1}{2} \cdot 12 \cdot 15 + \frac{1}{2} \cdot 5 \cdot 12 + \frac{1}{2} \cdot 24 \cdot 30 - \frac{1}{2} \cdot 10 \cdot 24 = 90 + 30 + 360 - 120 = 360.$$

Problem 16. [**13**] Suppose a and b are positive integers satisfying $a + b = 210$ and

$$6 \cdot \gcd^2(a, b) + \operatorname{lcm}^2(a, b) = 7ab.$$

Find the sum of all possible values of $|a - b|$.

Proposed by Rishabh Das

Answer. $\boxed{192}$

Solution. Let $a = ga_1$ and $b = gb_1$, where $g = \gcd(a, b)$. Then we have that

$$6g^2 + (ga_1b_1)^2 = 7g^2a_1b_1 \implies 6 + (a_1b_1)^2 = 7a_1b_1 \implies (a_1b_1 - 6)(a_1b_1 - 1) = 0.$$

If $a_1b_1 = 1$, then $a_1 = b_1 \implies a = b = 105$, which contributes nothing to our answer.

If $a_1b_1 = 6$, then either $a_1 = 6, b_1 = 1$ (or vice versa - but we only need to consider this case in our final answer), giving that $g = 30$ and $a = 180, b = 30$, or $a_1 = 3, b_1 = 2$, giving $g = 42$ and $a = 126, b = 84$.

This gives a final answer of $150 + 42 = 192$.

Problem 17. [**14**] Compute $\displaystyle\sum_{k=1}^{49} \sqrt{k - \sqrt{k^2 - 1}}$.

Proposed by Rishabh Das

Answer. $\boxed{5 + 3\sqrt{2}}$

Solution. Let $\sqrt{k - \sqrt{k^2 - 1}} = \sqrt{a} - \sqrt{b}$, for some a, b. Squaring both sides gives us

$$k - \sqrt{k^2 - 1} = (a + b) - 2\sqrt{ab}.$$

In an attempt to make a and b both nice, we try $a + b = k$ and $2\sqrt{ab} = \sqrt{k^2 - 1}$. Squaring both sides of this latter equation and simplifying, $ab = \frac{k^2-1}{4}$. Thus, we are looking to solve $a + b = k$ and $ab = \frac{k^2-1}{4}$. A bit of trial and error results in $\left\{\frac{k-1}{2}, \frac{k+1}{2}\right\}$ as a solution, so

$$\sqrt{k - \sqrt{k^2 - 1}} = \sqrt{\frac{k+1}{2}} - \sqrt{\frac{k-1}{2}}.$$

This means we want to compute

$$\sum_{k=1}^{49} \left(\sqrt{\frac{k+1}{2}} - \sqrt{\frac{k-1}{2}} \right).$$

Writing the terms out, we get

$$\left(\sqrt{\frac{2}{2}}-\sqrt{\frac{0}{2}}\right)+\left(\sqrt{\frac{3}{2}}-\sqrt{\frac{1}{2}}\right)+\left(\sqrt{\frac{4}{2}}-\sqrt{\frac{2}{2}}\right)+\left(\sqrt{\frac{5}{2}}-\sqrt{\frac{3}{2}}\right)+\cdots+\left(\sqrt{\frac{49}{2}}-\sqrt{\frac{47}{2}}\right)+\left(\sqrt{\frac{50}{2}}-\sqrt{\frac{48}{2}}\right).$$

Nearly every term cancels, and we're left with

$$\sqrt{\frac{50}{2}}+\sqrt{\frac{49}{2}}-\sqrt{\frac{1}{2}}-\sqrt{\frac{0}{2}}=5+\frac{7}{\sqrt{2}}-\frac{1}{\sqrt{2}}=5+\frac{6}{\sqrt{2}}=5+3\sqrt{2}.$$

Problem 18. [**14**] Triangle ABC has perimeter 5. If $\angle BAC=60^\circ$ and $AB^3+AC^3=12$, compute $AB\cdot AC$.

Proposed by Rishabh Das

Answer. $\boxed{\frac{5}{3}}$

Solution. Let $a=BC, b=CA$, and $c=AB$. Then by the Law of Cosines, $a^2=b^2-bc+c^2$. Additionally, we are given $b^3+c^3=12$. Factoring this gives

$$b^3+c^3=12 \implies (b+c)(b^2-bc+c^2)=12 \implies (b+c)(a^2)=12 \implies (5-a)(a^2)=12,$$

where we use the fact that $a+b+c=5$. We can guess $a=2$ is a root of this cubic relatively easily; if we wanted to, we could divide this cubic by $a-2$ and see the other two roots are either negative, or greater than 2.5, which can't happen by the triangle inequality, so $a=2$ is the only possibly value.

Now $b+c=3$ and $b^2-bc+c^2=4$. Squaring the first equation and subtracting the two gives $3bc=3^2-4 \implies bc=\frac{5}{3}$.

Problem 19. [**15**] If a, b, and c are positive reals that are not all the same satisfying

$$a^2+b^2+c=b^2+c^2+a=c^2+a^2+b,$$

then find the maximum possible value of abc.

Proposed by Rishabh Das

Answer. $\boxed{\frac{4}{27}}$

Solution. Just looking at the first two expressions,

$$a^2+c=c^2+a \implies a^2-c^2=a-c.$$

This means either $a-c=0$ or $a+c=1$. Similarly, either $b=c$ or $b+c=1$ and either $a=b$ or $a+b=1$.

We know that a, b, and c are not all the same. Thus, without loss of generality assume $a+b=1$. Then either $c=a$ or $c=1-a=b$. Thus, no matter what, two of a, b, and c will be the same, and the last one will be 1 minus this common value.

Our numbers are x, x, and $1-x$ in some order, and we want to maximize $x^2(1-x)$ for positive x. We can finish by taking a derivative and setting it equal to 0, but we can use AM-GM for a non-calculus solution:

$$x^2(1-x)=4\cdot\frac{x}{2}\cdot\frac{x}{2}\cdot(1-x)\le 4\left(\frac{\frac{x}{2}+\frac{x}{2}+(1-x)}{3}\right)^3=\frac{4}{27}.$$

Equality holds when $\frac{x}{2}=\frac{x}{2}=1-x$, or when $x=\frac{2}{3}$, so the maximum value of $x^2(1-x)$ is $\frac{4}{27}$.

Problem 20. **[Up to 28]** Welcome to **USAYNO**!

Instructions: Submit a string of 6 *letters corresponding to each statement: put* T *if you think the statement is true,* F *if you think it is false, and* X *if you do not wish to answer. You will receive* $\frac{(n+1)(n+2)}{2}$ *points for* n *correct answers, but you will receive zero points if any of the questions you choose to answer are incorrect. Note that this means if you submit "XXXXXX" you will get one point.*

(1) There exists a perfect square which is equal to the sum of the squares of five consecutive positive integers.

(2) A square $ABCD$ is completely covered by finitely many (possibly overlapping) disks. The sum of the radii of these disks must be at least $\frac{AC}{2}$.

(3) Suppose a, b, c, d, and e are integers satisfying

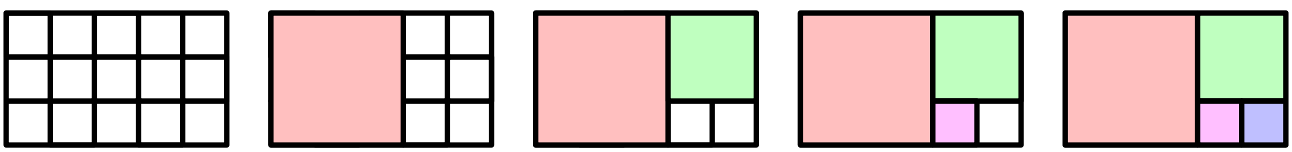
$$\begin{aligned} a+b &= c+d+e, \\ a^2+b^2 &= c^2+d^2+e^2, \text{ and} \\ a^3+b^3 &= c^3+d^3+e^3. \end{aligned}$$

Then $abcde = 0$.

(4) There exists an infinite set of positive integers S such that for any positive integers a and b, neither of which divides the other, at least one, but not all, of the integers $\gcd(a,b), a, b, \operatorname{lcm}(a,b)$ are elements of S.

(5) A rectangle has integer side lengths. We tile the rectangle with squares greedily; this means we first draw the largest possible square inside this rectangle (and if multiple exist, pick the top-left most one), which results in a smaller rectangle. We then repeat the process, tiling the rectangle with squares.

An example of a greedy tiling of a 3×5 rectangle is shown below.



This greedy method is the optimal way to tile any starting rectangle with squares (i.e. it uses the least number of squares out of any tiling of the rectangle with squares).

(6) Suppose O, H, and I are the circumcenter, orthocenter, and incenter of the acute, scalene $\triangle ABC$. The circumcircle of $\triangle OIH$ must pass through an even number of vertices of $\triangle ABC$.

Proposed by Rishabh Das and Vidur Jasuja

Answer. $\boxed{\text{FTTFFT}}$

Solution. The answer is FTTFFT.

(1) Of the five consecutive positive integers, let the middle one be a. Then the sum is

$$(a-2)^2 + (a-1)^2 + a^2 + (a+1)^2 + (a+2)^2 = 5a^2 + 10 = 5(a^2+2).$$

This is clearly a multiple of 5, so in order for the number to be a square we need it to be a multiple of 25. Thus, $5 \mid a^2 + 2$. However, we can check that $a^2 \equiv 3 \pmod 5$ has no solutions, so this number cannot be a square.

(2) Suppose $\Omega_1, \Omega_2, \ldots, \Omega_k$ are the circles that intersect AC, and have radius $r_1, r_2, \ldots, r_k$, respectively. Then

$$AC \leq \sum_{i=1}^{k} [\text{Length of intersection of } AC \text{ with } \Omega_i] \leq \sum_{i=1}^{k} 2 \cdot [\text{Radius of } \Omega_i] = \sum_{i=1}^{k} 2r_i,$$

which is at most 2 times the sum of the radii of all the circles. Thus, the sum of the radii of all the circles is at least $\frac{AC}{2}$.

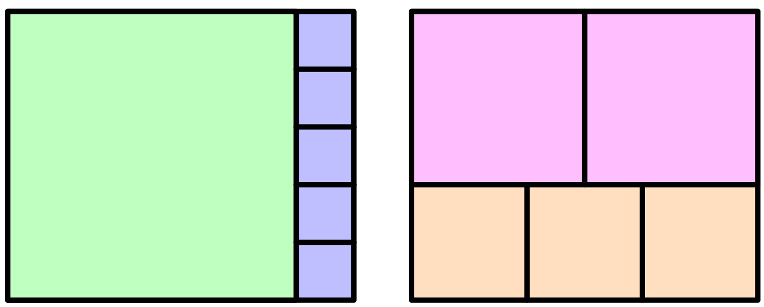
(3) Let $t = 0$. Then

$$a + b + t = c + d + e,$$
$$a^2 + b^2 + t^2 = c^2 + d^2 + e^2, \text{ and}$$
$$a^3 + b^3 + t^3 = c^3 + d^3 + e^3.$$

This means a, b, and t are the roots of a cubic , and c, d, and e are also roots of the same cubic. Thus, $\{a, b, t\} = \{c, d, e\}$. This means that one of c, d, and e is equal to $t = 0$, so $abcde = 0$.

(4) The idea is that we only need to consider integers of the form 2^a3^b. If we plot all such points in the coordinate plane, where (a, b) represents 2^a3^b, and color each point one of two colors, then the problem is equivalent to saying there's no monochromatic rectangle. This falls to a pigeonhole argument. Consider some arbitrary 3×7 intersection of three horizontal and seven vertical lines. At least four of these vertical lines will have the same majority color. Among these four vertical lines, it's impossible that we avoid a monochromatic rectangle; at least one pair of the lines will have two points of the majority color on the same horizontal lines. Therefore, the claim is false.

(5) The key is to force the greedy method to have a bunch of tiny squares; we can do this by letting one of the dimensions be 1 more than the other one. We can check the greedy method on 5×6 rectangle gives 6 squares, while we can actually tile it with only 5 squares.



(6) Since I is inside $\triangle ABC$, we know (OIH) cannot be the circumcircle of $\triangle ABC$. Thus, it's sufficient to check if (OIH) passes through exactly one vertex of $\triangle ABC$.

Suppose A is on (OIH), but B and C aren't. Since O and H are isogonal conjugates, $\angle OAI = \angle HAI$. Thus, $HI = IO$. But since $\angle OBI = \angle HBI$ and $\angle OCI = \angle HCI$, both B and C either lie on (OIH) or lie on the perpendicular bisector of OH. Since we have assumed both B and C are not on (OIH), they must both lie on the perpendicular bisector of OH. However, since $\triangle ABC$ is acute, both O and H lie inside $\triangle ABC$, a contradiction. (We also could've said I lies on BC as $IO = IH$, which is a contradiction.)

Problem 21. **[16]** Akash and Kimi are playing a game. Akash thinks of a polynomial $P(x)$ of degree 2021 with nonnegative integer coefficients, all at most 2021. A *move* is when Kimi gives Akash a real number r, and Akash tells Kimi $P(r)$. At most how many moves must Kimi make to determine the polynomial $P(x)$?

Proposed by Rishabh Das

Answer. $\boxed{1}$

Solution. We claim Kimi can win in one move; he clearly can't win in zero moves, so we just need to find a valid strategy for Kimi.

If Kimi gives Akash 2022, Akash will give back the number when the coefficients are concatenated together, in base 2022. Kimi can then find each digit in base 2022, and thus recover the polynomial. Thus, this strategy works, and the answer is 1.

Remark. In fact, if Kimi gives Akash 10000, then Akash will tell Kimi exactly the coefficients of P, with some 0s in between them.

There exist other strategies. For example, if Kimi gives Akash $\sqrt[2022]{3}$, Kimi wins in one move. (This strategy doesn't use any restriction on the coefficients besides them being rational!)

Problem 22. **[16]** Let a, b, c, d be positive real numbers such that

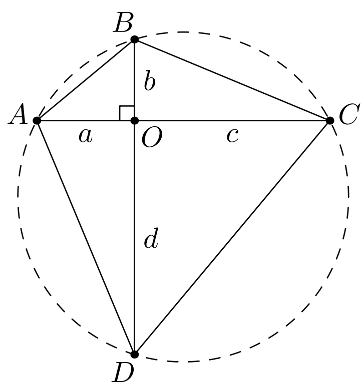
$$\begin{aligned} a^2 + b^2 &= 1, \\ b^2 + c^2 &= 16, \\ c^2 + d^2 &= 64, \text{ and} \\ ac &= bd. \end{aligned}$$

Compute $ab + bc + cd + da$.

Proposed by Vidur Jasuja

Answer. $\boxed{36}$

Solution 1. The sum of the first and third equations, minus the second one, gives $d^2 + a^2 = 49$. Construct four segments with a common endpoint, O, OA, OB, OC, and OD such that OA and OC are both perpendicular to OB and OD, and such that $OA = a$, $OB = b$, $OC = c$, and $OD = d$.



Four of our equations give $AB = 1$, $BC = 4$, $CD = 8$, and $DA = 7$. The equation we haven't used yet, $ac = bd$ means that $ABCD$ is cyclic by the converse of the power of a point. Then we have

$$ab + bc + cd + da = (a + c)(b + d) = 1 \cdot 8 + 4 \cdot 7 = 36$$

by Ptolemy's Theorem on cyclic quadrilateral $ABCD$.

Solution 2. The fourth equation tells us $d = \frac{ac}{b}$. Plugging this into the third one, we get

$$c^2 + \left(\frac{ac}{b}\right)^2 = 64 \implies \frac{c^2}{b^2} \cdot (a^2 + b^2) = 64.$$

Using the first equation makes this $\frac{c^2}{b^2} = 64$, or $\frac{c}{b} = 8$. Then by the second equation,

$$b^2 + c^2 = b^2 + (8b)^2 = 65b^2 = 16 \implies b = \frac{4}{\sqrt{65}}.$$

This then gives $c = \frac{32}{\sqrt{65}}$. The first equation now gives $a = \frac{7}{\sqrt{65}}$. Finally, the fourth equation gives $d = \frac{56}{\sqrt{65}}$. Now we can compute

$$ab + bc + cd + da = (a + c)(b + d) = \left(\frac{7 + 32}{\sqrt{65}}\right)\left(\frac{4 + 56}{\sqrt{65}}\right) = \frac{39 \cdot 60}{65} = 36.$$

Problem 23. **[17]** A positive integer n is *similvisible* if it has only single-digit prime factors and $4n, 5n, 6n$, and $7n$ have the same number of positive integer divisors. Compute the sum of the reciprocals of all similvisible integers.

Proposed by Vidur Jasuja

Answer. $\boxed{\frac{70}{1259}}$

Solution. We may write $n = 2^a 3^b 5^c 7^d$. The number of divisors of $4n$ is

$$(a+3)(b+1)(c+1)(d+1),$$

the number of divisors of $5n$ is

$$(a+1)(b+1)(c+2)(d+1),$$

the number of divisors of $6n$ is

$$(a+2)(b+2)(c+1)(d+1),$$

and the number of divisors of $7n$ is

$$(a+1)(b+1)(c+1)(d+2).$$

Comparing the second and fourth equations gives that $c = d$. Comparing the first and second gives that $(a+3)(c+1) = (a+1)(c+2)$, which gives that $a = 2c+1$. Finally, comparing the first and third equation gives that $(a+3)(b+1) = (a+2)(b+2)$, so $b = a+1$.

This means that all similvisible numbers are of the form

$$2^{2c+1}3^{2c+2}5^c7^c = 18 \cdot 1260^c$$

So, we want to compute

$$\sum_{n=0}^{\infty} \frac{1}{18} \cdot \left(\frac{1}{1260}\right)^c = \frac{\frac{1}{18}}{1 - \frac{1}{1260}} = \frac{70}{1259}.$$

Problem 24. **[17]** Suppose $(\sigma_1, \sigma_2, \ldots, \sigma_{2021})$ is a permutation of $(1, 2, 3, \ldots, 2021)$. Suppose, across all such permutations, m is the minimum value of the expression

$$|\sigma_2 - \sigma_1| + |\sigma_3 - \sigma_2| + \cdots + |\sigma_{2021} - \sigma_{2020}| + |\sigma_1 - \sigma_{2021}|$$

and suppose n is the number of such permutations such that the given expression is equal to m. Compute n.

Proposed by Vidur Jasuja

Answer. $\boxed{2021 \cdot 2^{2019}}$

Solution. We claim that $m = 4040$. To prove this, we will make use of the fact that $|a| + |b| \geq |a+b|$, and that equality holds when a and b have the same sign. Suppose, without loss of generality, that $\sigma_1 = 1$. Suppose that $\sigma_k = 2021$, for some k. Then, we know that

$$|\sigma_2 - \sigma_1| + |\sigma_3 - \sigma_2| + \cdots + |\sigma_k - \sigma_{k-1}| \geq |\sigma_k - \sigma_1| = 2020$$

and

$$|\sigma_{k+1} - \sigma_k| + |\sigma_{k+2} - \sigma_{k+1}| + \cdots + |\sigma_1 - \sigma_{2021}| \geq |\sigma_1 - \sigma k| = 2020.$$

Summing these two inequalities gives the desired bound of 4040. This will be achieved when all terms in each inequality have the same sign. That is, the permutation must increase from $\sigma_1 = 1$ to $\sigma_k = 2021$, and then decrease in the other part of the permutation.

To count this, note that for each integer from 2 to 2020, we can put it in either the increasing or decreasing part of the permutation, and once we're done, the permutation is fixed. So there are 2^{2019} ways to place these integers.

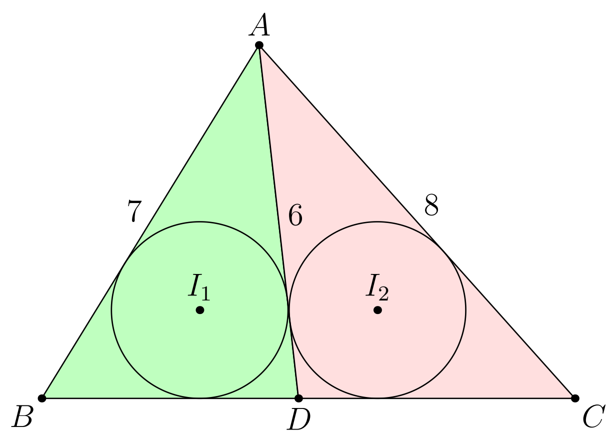
Finally, we must multiply by 2021 to account for where we place 1, as it can go anywhere, rather than just σ_1. This yields the final answer of $2021 \cdot 2^{2019}$.

Problem 25. **[18]** Let triangle ABC be such that $AB = 7$ and $AC = 8$. There exists a point D on segment $\overline{BC}$ such that $AD = 6$ and the inradius of triangle ABD is equal to the inradius of triangle ACD. Find BC.

Proposed by Rishabh Das and Vidur Jasuja

Answer. $\boxed{9}$

Solution.



The inradius of $\triangle ABD$ is equal to

$$\frac{[ABD]}{s_{ABD}},$$

and the inradius of $\triangle ACD$ is equal to

$$\frac{[ACD]}{s_{ACD}},$$

where $[\bullet]$ denotes the area of a polygon and $s_\bullet$ is the semiperimeter of a polygon. We need these two values to be equal, so

$$\frac{[ABD]}{[ACD]} = \frac{s_{ABD}}{s_{ACD}}.$$

Triangles ABD and ACD share the same altitude from A, so the ratio of their areas is the ratio of their bases, which is just $\frac{BD}{CD}$. Thus,

$$\frac{[ABD]}{[ACD]} = \frac{BD}{CD} = \frac{s_{ABD}}{s_{ACD}} = \frac{\frac{6+7+BD}{2}}{\frac{6+8+CD}{2}} = \frac{13+BD}{14+CD}.$$

By the Baseball Theorem, we can subtract BD from the numerator of the fraction on the right and CD from the denominator on the right, so this common value is just $\frac{13}{14}$. Let $BD = 13x$ and $CD = 14x$.

By Stewart's theorem, we have

$$\begin{aligned}
13x \cdot 14x \cdot 27x + 6^2 \cdot 27x &= 7^2 \cdot 14x + 8^2 \cdot 13x \\
13 \cdot 14 \cdot 27 \cdot x^2 &= 64 \cdot 13 + 49 \cdot 14 - 36 \cdot 27 \\
13 \cdot 14 \cdot 27 \cdot x^2 &= 832 + 686 - 972 = 546 \\
27 \cdot x^2 &= 3 \\
x &= \frac{1}{3}.
\end{aligned}$$

Now $BC = 27x = 27 \cdot \frac{1}{3} = 9$.

Problem 26. [18] Let $m = 2^4 \cdot 3^4$. Suppose k is a randomly selected integer from 1 to m, inclusive. Let ℓ be the expected value of $\log_{10}(\gcd(k, m))$. Find the number of not necessarily distinct prime factors of $10^{\ell m}$. (For example, $12 = 2^2 \cdot 3$ has 3 not necessarily distinct prime factors.)

Proposed by Paul Gutkovich

Answer. $\boxed{1855}$

Solution. By definition, we have

$$\ell = \frac{1}{m}\sum_{k=1}^{m} \log_{10}(\gcd(k,m)) = \frac{1}{m}\log_{10}\left(\prod_{k=1}^{m}\gcd(k,m)\right).$$

Then $10^{\ell m} = \prod_{k=1}^{m}\gcd(k,m)$, so we just need to compute this.

Note that $\gcd(k, 2^4) \cdot \gcd(k, 3^4) = \gcd(k, 2^4 \cdot 3^4) = \gcd(k, m)$, so we can rewrite our product as

$$\prod_{k=1}^{2^4\cdot 3^4}\gcd(k,2^4)\cdot\gcd(k,3^4) = \left(\prod_{k=1}^{2^4\cdot 3^4}\gcd(k,2^4)\right)\cdot\left(\prod_{k=1}^{2^4\cdot 3^4}\gcd(k,3^4)\right) = \left(\prod_{k=1}^{2^4}\gcd(k,2^4)\right)^{3^4}\cdot\left(\prod_{k=1}^{3^4}\gcd(k,3^4)\right)^{2^4},$$

as k ranges over the residues mod $2^4 = 16$ exactly $3^4 = 81$ times, and vice versa. We can compute $\prod_{k=1}^{2^4}\gcd(k,2^4) = 1^8 \cdot 2^4 \cdot 4^2 \cdot 8^1 \cdot 16^1 = 2^{15}$ and $\prod_{k=1}^{3^4}\gcd(k,3^4) = 1^{54} \cdot 3^{18} \cdot 9^6 \cdot 27^2 \cdot 81^1 = 3^{40}$. Thus, our final number is equal to $2^{15\cdot 3^4}3^{40\cdot 2^4} = 2^{1215}3^{640}$. This means the final answer is $1215 + 640 = 1855$.

Problem 27. [19] Suppose c is a real number such that when the roots of $x^3 - 3x^2 + 12x + c$ are plotted in the complex plane, they form a non-degenerate triangle with orthocenter at the origin. Compute c.

Proposed by Vidur Jasuja

Answer. $\boxed{260}$

Solution. Evidently, our triangle is symmetric about the real axis, by the conjugate root theorem. It will have one vertex A on the axis, and two more vertices not on the axis, its complex roots B and C.

Let $D = d$ be the foot from A to $\overline{BC}$. Then we can write $A = d + a$, $B = d + bi$, $C = d - bi$, for some reals a and b. Now, since $AC \perp BH$, we have that $a + bi$ and $d + bi$ are perpendicular, so then their quotient is imaginary. Since

$$\frac{a+bi}{d+bi} = \frac{(a+bi)(d-bi)}{d^2+b^2} = \frac{(ad+b^2)+(db-ab)i}{d^2+b^2}$$

has real part zero, we find that $ad = -b^2$.

Now, we will apply Vieta's formulas. We have that

$$(d+a) + (d+bi) + (d-bi) = 3d + a = 3$$

and

$$(d+a)(d+bi) + (d+a)(d-bi) + (d+bi)(d-bi) = 2d^2 + 2ad + d^2 + b^2 = 3d^2 + ad = 12.$$

Dividing this equation by the first one, we have $d = 4$, so $a = -9$. Then, our answer is

$$-(d+a)(d+bi)(d-bi) = -(d+a)(d^2+b^2) = -(d+a)(d^2-ad) = -(-5)\cdot 52 = 260.$$

Problem 28. [19] Alex writes the ordered pair $(1,0)$ on a chalkboard. Every minute, he randomly and uniformly chooses two integers c and d, between 1 and 5, inclusive. Then, if at that time he has the ordered pair (a,b) on the board, he erases it and writes the ordered pair $(ac+bd, ad+bc)$ on the board. Find the expected number of minutes it will take for both of the numbers in his ordered pair to be divisible by 5.

Proposed by Vidur Jasuja

Answer. $\boxed{\frac{65}{9}}$

Solution 1. The crux of this problem rests on considering the sum and difference of the two numbers of the board. Both numbers on the board will be multiples of 5 iff their sum and difference are multiples of 5. That is, $ac+bd+ad+bc = (a+b)(c+d)$, and $ac+bd-ad-bc = (a-b)(c-d)$, are multiples of 5. This means that when we are done, we must have selected at least one pair (c,d) at some point with sum a multiple of 5 and one with difference a multiple of 5. Note that with $(5,5)$ we achieve both of these, and are thus instantly done.

To proceed, we present two approaches. The first is using states. Let E_0 denote the expected number of minutes that it will take Alex in total, and let E_1 denote the expected number of minutes it will take Alex if he has at some point selected a pair with sum a multiple of 5 but not difference, or vice versa (note that this is allowed by symmetry). Then

$$E_0 = \frac{16}{25}(E_0+1) + \frac{8}{25}(E_1+1) + \frac{1}{25}(1),$$

and

$$E_1 = \frac{4}{5}(E_1+1) + \frac{1}{5}(1) \implies \frac{1}{5}E_1 = 1 \implies E_1 = 5.$$

Plugging this into the first equation gives

$$E_0 = \frac{16}{25}E_0 + \frac{65}{25} \implies E_0 = \frac{65}{9}.$$

Solution 2. We present an alternate finish to the above one. We require the same initial observations. Let X be a random variable that is equal to the number of minutes until both numbers in the ordered pair are divisible by 5. We use the fact that if X only takes on nonnegative integer values, then

$$\mathbb{E}[X] = \sum_{k=1}^{\infty} k\,\mathrm{Prob}(X=k) = \sum_{k=1}^{\infty} \mathrm{Prob}(X \ge k).$$

(Note that if X didn't take only nonnegative integer values, we would have to integrate $\mathrm{Prob}(X \ge k)$, but a similar method would work.) We can compute this with PIE: each of the two numbers in the ordered pair has probability $\left(\frac{4}{5}\right)^{k-1}$ of not being a multiple of 5 after $k-1$ moves, and they have probability $\left(\frac{16}{25}\right)^{k-1}$ of both not being multiples of 5 after $k-1$ moves. Thus, the probability $X \ge k$ is

$$2\left(\frac{4}{5}\right)^{k-1} - \left(\frac{16}{25}\right)^{k-1}.$$

Summing over all $k \ge 1$, we get

$$\mathbb{E}[X] = \sum_{k=1}^{\infty} 2\left(\frac{4}{5}\right)^{k-1} - \left(\frac{16}{25}\right)^{k-1} = 2\cdot\frac{1}{1-4/5} - \frac{1}{1-16/25} = 10 - \frac{25}{9} = \frac{65}{9}.$$

During the contest, the problem mistakenly had the pair $(1,1)$ *rather than* $(1,0)$*. This meant the difference was always divisible by* 5*, making the answer just* 5*. This was marked correct during the contest.*

Problem 29. **[20]** Stan the Cat is a 2021-dimensional creature, playing with 2021-dimensional hyperspherical marbles $\Omega_1, \Omega_2, \ldots, \Omega_{2023}$. The radii of $\Omega_1, \Omega_2, \ldots, \Omega_{2021}$ are 1, while the radii of Ω_{2022} and Ω_{2023} are r. Stan then arranges his marbles such that they are all pairwise externally tangent. Find r.

Proposed by Rishabh Das

Answer. $\boxed{\frac{2019}{4042}}$

Solution 1. We solve the problem for general $n \geq 3$, rather than 2021. (It may be helpful to visualize the diagram for $n = 3$.)

Scale the problem down by $\sqrt{2}$; we will multiply our final radius by $\sqrt{2}$ at the end. Then the radius of Ω_i is $\frac{\sqrt{2}}{2}$ for $1 \leq i \leq n$. Let the center of Ω_i be O_i.

Place O_i at the point where all coordinates are 0, except for the ith one, which is a 1, for $1 \leq i \leq n$. Then O_{n+1} and O_{n+2} have centers that have coordinates that are all equal. This means that both of them are tangent to the hyperplane $x_1 + x_2 + \cdots + x_n = 1$ at $\left(\frac{1}{n}, \frac{1}{n}, \frac{1}{n}, \ldots, \frac{1}{n}\right) = X$. Now $O_{n+1}O_1 = \frac{\sqrt{2}}{2} + r$, $XO_{n+1} = r$, and O_1X is the distance from $(1, 0, 0, \ldots, 0)$ to $\left(\frac{1}{n}, \frac{1}{n}, \frac{1}{n}, \ldots, \frac{1}{n}\right)$, which is

$$\sqrt{(n-1)\left(\frac{1}{n}\right)^2 + \left(\frac{n-1}{n}\right)^2} = \sqrt{\frac{(n-1)n}{n^2}} = \sqrt{\frac{n-1}{n}}.$$

We also have $\triangle O_1O_{n+1}X$ is a right triangle with right angle at X, so by the Pythagorean Theorem,

$$XO_1^2 + XO_{n+1}^2 = O_1O_{n+1}^2 \implies \frac{n-1}{n} + r^2 = \left(r + \frac{\sqrt{2}}{2}\right)^2 \implies r\sqrt{2} = \frac{n-1}{n} - \frac{1}{2} = \frac{n-2}{2n}.$$

Since we're looking for the radius of Ω_{n+1} scaled up by $\sqrt{2}$, the final radius is $r\sqrt{2} = \frac{n-2}{2n}$.

With $n = 2021$, the answer is $\frac{2019}{4042}$.

Solution 2. Set up the coordinate plane similar to before. Perform an inversion about the n-dimensional hypersphere centered at X fixing Ω_i for $1 \leq i \leq n$. The radius of this inversion is

$$\sqrt{XO_1^2 - [\text{radius of } \Omega_1]^2} = \sqrt{XO_1^2 - \frac{1}{2}}.$$

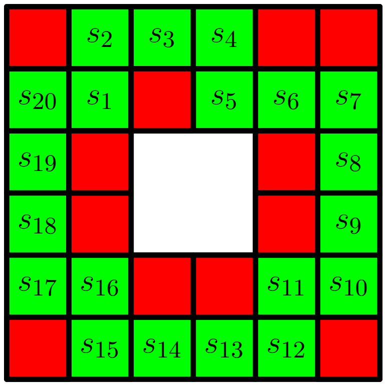
The point diametrically across from X on Ω_{n+1} maps to the foot of X to a hyperplane tangent to Ω_i for $1 \leq i \leq n$. The distance from X to this point is just equal to the radius of Ω_i for $1 \leq i \leq n$, which is $\frac{\sqrt{2}}{2}$. Thus,

$$\frac{n-2}{2n} = (2r) \cdot \frac{\sqrt{2}}{2} = r\sqrt{2},$$

so the final answer is $\frac{n-2}{2n}$.

Problem 30. **[20]** A 32×32 grid has the middle 28×28 grid cut out, forming a "donut" of width 2. From this donut, a subset of cells of size m is called *m-cool* if they can be labelled $s_1, s_2, \ldots, s_m$ such that s_i and s_j share exactly one side if and only if $i - j \equiv \pm 1 \pmod{m}$. Let M be the maximum positive integer such than an M-cool set exists, and suppose there are N distinct M-cool sets. Compute N.

The green cells below are an example of a 20-cool set on a smaller donut (a 2×2 removed from a 6×6).



Proposed by Rishabh Das

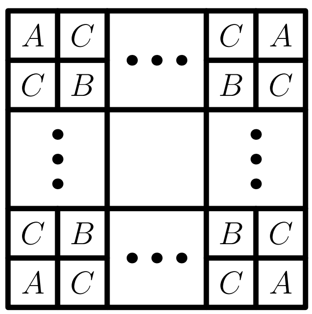
Answer. $\boxed{51527}$

Solution. First, we claim $M = 180$. Call a square in our subset *green* and other cells *red* (as in our above example).

Every 2×2 square can contain at most 3 green cells. Suppose a 2×2 square had 4 green cells. Then two of them in opposite corners are both adjacent to the other two, which contradicts the "if and only if" portion. Thus, after partitioning the donut into $\frac{32^2-28^2}{4} = \frac{60 \cdot 4}{4} = 60$ 2×2 squares, we see there are at most $60 \cdot 3 = 180$ green cells.

As a consequence of this equality case, when we partition the donut into 2×2 squares, all of them will have exactly 3 of the green cells, and one red cell. We will perform casework on the 4 corner 2×2 squares.

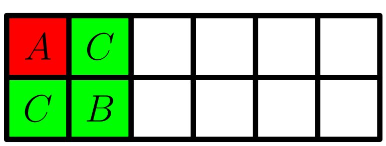
In each of the four corners, label the cells as follows.



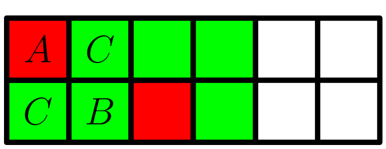
We claim that the cells labeled C cannot be red. If they were, then a cell labeled A would only be adjacent to one green cell, a contradiction. Thus, these corner 2×2 will have either A or B as their red cell.

Note that once we have filled in which cell is red in the corners, the four "edge" parts act independently of each other. Thus, we will look at two adjacent corners, and do casework on the number of possibilities based on which cells in the corners are red.

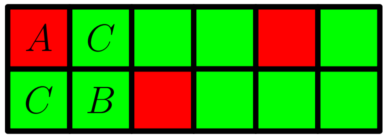
First consider the case where one of two have an A cell as red.



The cell labeled with B is adjacent to two green cells already, so the last cell it's adjacent to must be red. Then, since this can be the only red in its 2×2 square, we can fill in the next 4 squares.

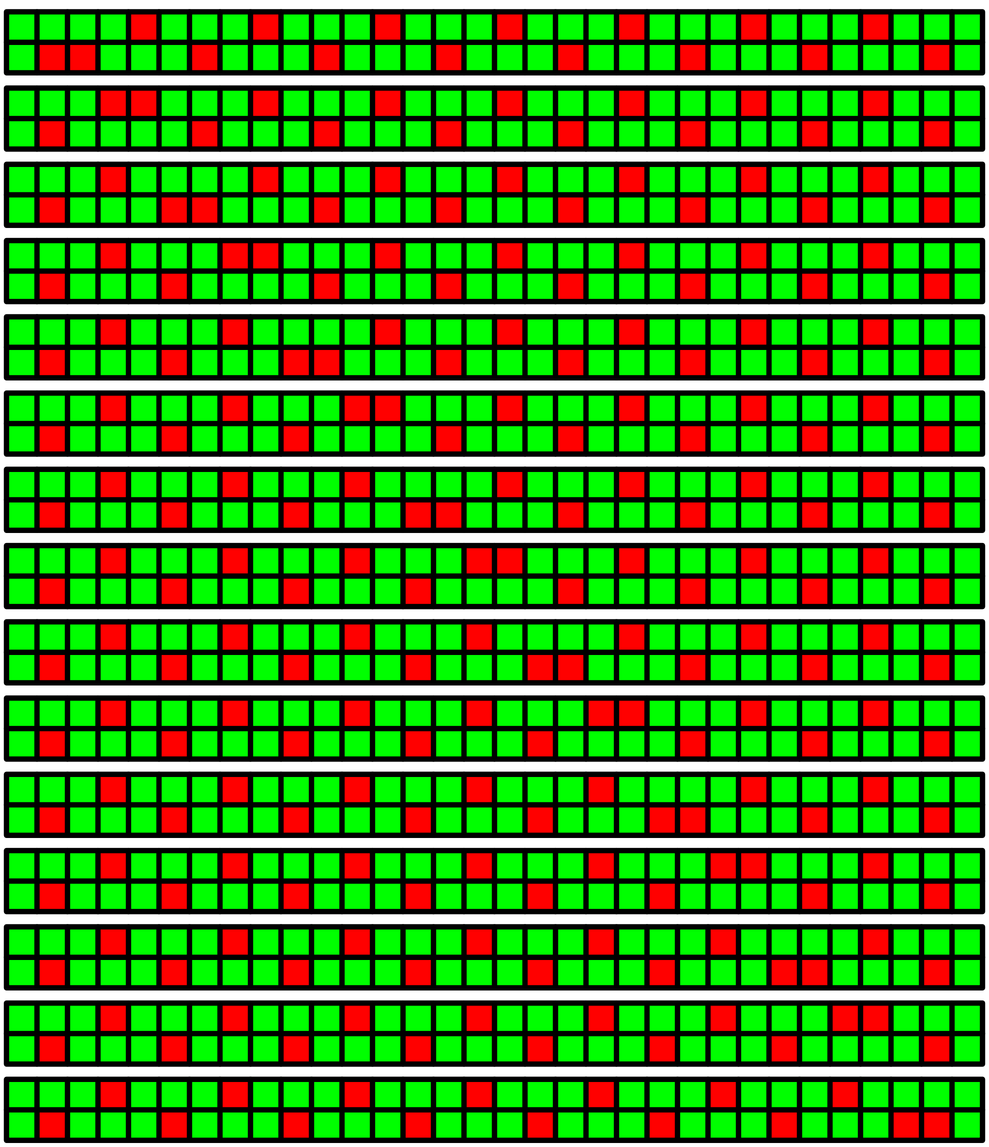


Repeating this argument once more, we get the following:



However, then we can repeat this argument since we are in virtually the same situation we started with, alternating between the two types of 2×2 blocks. Since there are an even number of 2×2 blocks in a row (namely 16 of them), the 2×2 in the adjacent corners must have the B cell colored as red. Thus, so far we've determined that if two adjacent corners both have their A cell colored red, there are 0 ways to color the squares between them, and if one has their A cell colored red and the other has their B cell colored red, there is 1 way to color the squares between them. We are left to do the case where both have their B square colored red.

Without loss of generality, assume we're working on the top row, i.e. the top two corners have their B as their red cell. Work from the left to the right. Note that in each 2×2 cell, starting from the left, if a cell on the left is red, then the rest of the coloring is determined. (This is basically how we did previous cases.) Also note that if a cell on the right is red, then the next 2×2 has 2 cases for the red cell; one on the right (in which case we're done by what we just did) or one on the left, in which case we just repeat the argument. We need to flip from being on the right to being on the left at some point, as we end in the top-right corner with a B cell colored red, which is on the left of its 2×2 square. There are 15 places where we can flip, so there are 15 choices for the coloring. They are all displayed below.



With this, we can compute our answer.

If the corners have no red A cells, there are $15^4 = 50625$ ways to color the squares.

If there is 1 red A cell, there are 4 ways to choose which corner has the red A cell, and 15^2 ways to color the remaining cells, as the long edges containing the red A cell are fixed. Thus, there are $4 \cdot 15^2 = 900$ ways for this case to happen.

Finally, if there are 2 red A cells, there are 2 ways to choose which corners have the red cells, as they must be opposite each other. Each of these cases gives 1 way to color the rest of the donut, since each of these red A cells are adjacent to two red B cells. Thus, this case gives $2 \cdot 1 = 2$.

Thus, the final answer is $50625 + 900 + 2 = 51527$.