NYCTC Spring 2024 Results and Solutions

NYCTC Writers

June 28, 2024

Contents

1 Results

There were 37 teams at NYCTC, amounting to about 125 competitors, present on Friday — just a little shy of the winter contest. Notably, over a quarter of those people signed up within 24 hours of the contest.

1.1 Acknowledgements

This contest was written by Aditya Pahuja, Daniel Potievsky, Noam Pasman, and Steven Lou. Our thanks go to Calvin Zhang, Mikayla Lin, and Josiah Moltz for proctoring, and to Rishabh Das, Boyan Litchev, Cathy Deng, Katherine Li, and Michael Aram, for testsolving and providing extremely useful feedback on the problems.

1.2 Results

The top two teams were neck-and-neck for the entire contest. Luckily, andruwu came out on top; otherwise, Aditya would've had to buy a watermelon for =B2. The top three teams are enumerated below:

- **1st Place.** Team andruwu, consisting of Anastasia Lee, Andrew Li, Ashley Zhu, and Ian Buchanan, with 299.58 points.
- **2nd Place.** Team =B2, consisting of Gabe Levin, James Papaelias, Kylan Huang, and Kyle Wu, with 282.42 points.
- **3rd Place.** Team Twilight Sparkle, consisting of Dieter Yang, Harrison Cai, Max Shepard, and Steven Mao, with 246.58 points.

Congratulations to these teams for a very strong performance! The full leaderboard can be accessed at this link. Note that second place says andruwu; this is because the team name =B2 was submitted with the intent to behave like Google Sheets code that copies the content of cell B2, i.e. the name of the first place team.

Finally, thank you to everyone who participated! We hope you enjoyed the problems, and we hope that you'll return for the next generation of this contest in the winter, whether as students or as alumni.

1.3 Solve Rates

Problem	Solve %	Avg Pts Earned
1	75.68	4.54
2	86.49	5.19
3	43.24	3.03
4	91.89	6.43
5	83.78	6.7
6	97.3	7.78
7	81.08	7.3
8	89.19	8.03
9	86.49	8.65
10	—	3.86
11	94.59	10.41
12	48.65	5.35
13	70.27	8.43
14	56.76	6.81
15	51.35	6.68
16	67.57	8.78
17	64.86	9.08
18	40.54	5.68
19	16.22	2.43
20	—	2.59
21	35.14	5.62
22	27.03	4.32
23	27.03	4.59
24	32.43	5.51
25	21.62	3.89
26	21.62	3.89
27	2.7	0.51
28	0	0
29	0	0
30	0	0

More visually:

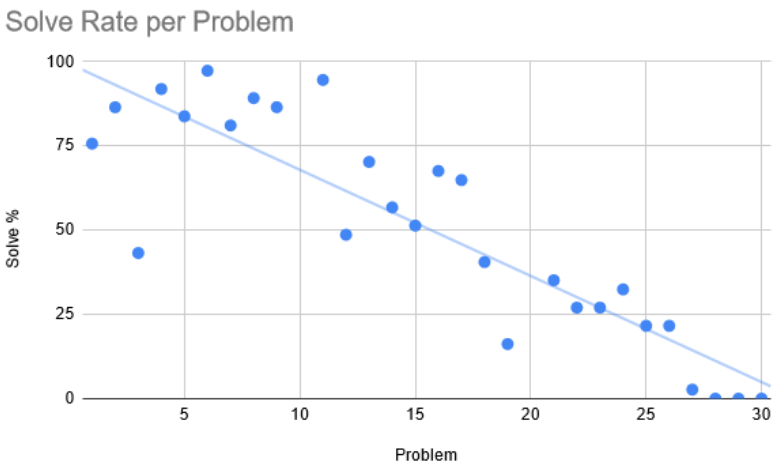


Figure 1: The difficulty trends in the correct direction, but fluctuates a lot.

Additionally, the average team solved 14.14 problems, with a standard deviation of 5.06 (excluding the minigame and USAYNO), and the median number of solves was 13.

2 Solutions

Problem 1. [**6**] Compute

$$(2+0+2+4)(2!+0!+2!-4^2)(2^2-0!+2^2+4^2).$$

Proposed by Noam Pasman

Solution. The product simplifies as

$$8\cdot(-11)\cdot 23 = \boxed{-2024}.$$ □

Problem 2. [**6**] The answer to this problem can be expressed as $\frac{a}{b}$, where a and b are relatively prime positive integers. Compute $2\sqrt{ab}$.

Proposed by Aditya Pahuja

Solution. We are given that, on one hand, the answer is the rational $\frac{a}{b}$, and on the other hand, the square root of the integer $4ab$. The square root of an integer being rational implies that the square root is an integer, meaning $\frac{a}{b}$ is an integer. Since b divides a and $\gcd(a,b)=1$, we must have $b=1$, so that $2\sqrt{a}=a$. This forces $a=\frac{a}{b}=\boxed{4}$. □

Problem 3. [**7**] Find the positive integer $n<100$ for which the ratio of n to the sum of its divisors is maximal.

Proposed by Aditya Pahuja

Solution. Observe that n is always less than or equal to the sum of its divisors, with equality if and only if $n=1$, since otherwise the sum of the divisors of n has a lower bound of $n+1$. Therefore, the maximum ratio of n to the sum of its divisors is 1, when $\boxed{n=1}$. □

> **Remark.** Over half the participating teams submitted the incorrect answer 97.

Problem 4. [**7**] Compute $\log_{32} 81\cdot\log_6 7\cdot\log_3 36\cdot\log_{343} 2$.

Proposed by Noam Pasman

Solution. Using the change of base formula, we get

$$\frac{\log 81}{\log 32}\cdot\frac{\log 7}{\log 6}\cdot\frac{\log 36}{\log 3}\cdot\frac{\log 2}{\log 343} = \left(\frac{4}{5}\cdot\frac{\log 3}{\log 2}\right)\cdot\frac{\log 7}{\log 6}\cdot\left(2\cdot\frac{\log 6}{\log 3}\right)\cdot\left(\frac{1}{3}\cdot\frac{\log 2}{\log 7}\right) = \boxed{\frac{8}{15}}.$$ □

Problem 5. [**8**] Let $\triangle ABC$ have area 256. The midpoints of $\overline{AB}$ and $\overline{AC}$ are D and E, respectively. Points F and G are on segments $\overline{AB}$ and $\overline{AC}$ respectively such that $\overline{FG}$ is parallel to $\overline{BC}$ and the lengths of $\overline{DE}$, $\overline{FG}$, and $\overline{BC}$ form a geometric sequence in some order. What is the sum of the possible areas of $\triangle AFG$?

Proposed by Aditya Pahuja

Solution. We already know that $\overline{DE}$ and $\overline{FG}$ are both shorter than $\overline{BC}$ (noting that F and G are specified to lie on *segments* $\overline{AB}$ and $\overline{AC}$ respectively), so we only have to deal with two cases:

$$\frac{DE}{FG}=\frac{FG}{BC} \quad\text{or}\quad \frac{FG}{DE}=\frac{DE}{BC}.$$

Write $DE=\frac{BC}{2}$; then, the first case gives $\frac{FG}{BC}=\frac{1}{\sqrt{2}}$, and the second case gives $\frac{FG}{BC}=\frac{1}{4}$. This means that the area of $\triangle AFG$ is

$$[AFG]=[ABC]\cdot\left(\frac{FG}{BC}\right)^2=\frac{256}{2} \text{ or } \frac{256}{16},$$

so the sum of the possible areas of the triangle is $128+16=\boxed{144}$. □

Problem 6. [**8**] Which positive integer is four times a prime number and one less than a perfect cube?

Proposed by Aditya Pahuja

Solution. Let n be the integer. Then, for some prime p and integer k,

$$n = 4p = k^3 - 1.$$

Since $k^3 - 1 = (k-1)(k^2+k+1)$ and the larger factor (k^2+k+1) is odd, we can deduce that $k - 1 = 4$ is required, forcing $k = 5$. As expected, $k^2 + k + 1 = 31$ is prime, so $\boxed{n = 124}$ works. □

Problem 7. [**9**] Quadrilateral $ABCD$ has $AB = 3$, $BD = 4$, and $CD = 5$. Find the maximum possible area of $ABCD$.

Proposed by Noam Pasman

Solution. We can think of $ABCD$ as two triangles $\triangle ABD$ and $\triangle BCD$ glued together at $\overline{BD}$. Then, the maximum of $[ABD]$ is $\frac{3\cdot4}{2}$, since the height from A has length at most 3, and similarly, the maximum of $[BCD]$ is $\frac{5\cdot4}{2}$ since the height from C has length at most 5. This maximum is of course achieved when $\angle ABD = \angle BDC = 90^\circ$, so the answer is $6 + 10 = \boxed{16}$. □

Problem 8. [**9**] Compute the number of positive integers n such that the decimal representations of n and $2n$ each have exactly one odd digit and one even digit.

Proposed by Aditya Pahuja

Solution. Let $n = 10a + b$ for digits a and b. Obviously $a < 5$ is necessary because $2n$ has two digits, and $b \geq 5$ is necessary because $b < 5$ implies that $2n = 10(2a) + (2b)$ has two even digits. The two conditions above are sufficient for $2n$ to have exactly one odd and one even digit, since

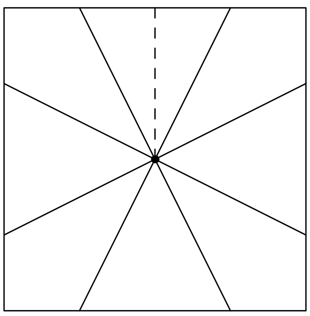
$$2n = 10(2a+1) + (2b-10)$$

has an odd digit of $2a + 1$ and an even digit of $2b - 10$. Noting also that a and b must have different parities in order for the condition on n to be satisfied, the total count is $2 \cdot 5 = \boxed{10}$ since, for each b, there are exactly two valid choices of a with the opposite parity. □

Problem 9. [**10**] Four distinct congruent segments are drawn through the center of a square $\mathcal{S}$ such that the endpoints of each segment are on the perimeter of the square. If the segments partition $\mathcal{S}$ into eight pieces of area 18, what is the length of each of these segments?

Proposed by Aditya Pahuja

Solution. The segments split $\mathcal{S}$ into four quadrilaterals and four triangles.



Letting s be the side length of the square, we know that $s^2 = 8 \cdot 18 = 144$, so $s = 12$. Then, each triangle has height 6, since that is the distance from the center to each side, so each triangle must have a base of length 6 as well. This means the distance from the center of the square to one of the endpoints of a segment is

$$\sqrt{3^2 + 6^2} = 3\sqrt{5},$$

and then the length of the entire segment is twice this length, which is $2 \cdot 3\sqrt{5} = \boxed{6\sqrt{5}}$. □

Problem 10. [Up to 10] Welcome to **Proof by Democracy**, the minigame where you (pl.) get to decide the truth of some famous open problems!

Instructions: The following six statements are currently open: nobody knows whether they are true or false. Submit a string of 6 letters in which the kth letter is Y if you think statement k is true and N if you think it is false. If the proportion of teams that have the same answer as you for statement k is x_k, then you will receive $10\sqrt[3]{x_1x_2x_3x_4x_5x_6}$ points.

(a) (Fortune's conjecture) Denote the sequence of primes in increasing order by p_1, p_2, p_3, $\ldots$. For a given positive integer n, let m be the smallest integer greater than 1 such that $p_1p_2\cdots p_n+m$ is prime. Then, m must also be prime.

(b) (Euler brick) There exists an $a\times b\times c$ rectangular prism in which the distance between any two vertices is an integer.

(c) (No-three-in-line problem) Let n be an integer greater than 1. It is always possible to place $2n$ pawns on an $n\times n$ chessboard such that no three are collinear.

(d) (Brocard's problem) There exists an integer $n>7$ for which $n!+1$ is a square.

(e) (Erdős-Straus conjecture) For every $n\geq 2$, there are positive integers x, y, z such that

$$\frac{4}{n}=\frac{1}{x}+\frac{1}{y}+\frac{1}{z}.$$

(f) (Inscribed square problem) On every closed curve in the plane, one can find four points that form a square.

Proposed by Aditya Pahuja

The optimal answer was $\boxed{\text{YYYYYY}}$.

Team Name	Submission	Score	Team Name	Submission	Score
Twilight Sparkle	YYYYYY	5.58	=B2	YNYYYN	2.42
jacob's pickles	YYYYYY	5.58	yee jingles wen yesley jiangles	NNYNYY	2.27
baz	YYYNYY	4.91	🍉	NYNYNY	1.43
I hate swimming	YYYYYY	5.58	Gucci Gang	YNYNYY	3.69
Aunt Em and Uncle Henry	NYYNNY	1.75	Christopher Shen's Right Foot	YYYYYY	5.58
Calculation Wizards	YYYYYY	5.58	Pinkie Pie	YYYYYY	5.58
Danger Third Rail	YYYYYY	5.58	8:06	YYYYYY	5.58
Applejack	YYNYNY	2.32	The team above us is cheating	YYYYYY	5.58
roggenrola	YNYNYY	3.69	Team chofulee	YYNNYN	2.04
Our Mogussy Shrill Filthy Bump	NNYNYY	2.27	mma	YYNYNY	2.32
Ethan Zhang	YYNYYN	2.32	A long, cold, lonely winter	YNNYYY	3.01
1/2 Asian	NNYNYY	2.27	eepy kittens :3	YYYYYY	5.58
liam x bryce canon	NYNNYN	1.26	Zesty lemons	YNYNYN	2.13
The Collegiate School	NNNNNN	0.55	The Lance-Phoenixes	YYYYYY	5.58
Rainbow Dash	YYNNYY	3.53	Return of the Mandelbrot set	YNYNYY	3.69
andruwu	YYYYYY	5.58	Yes	YYYYYY	5.58
gonk	YYYYYY	5.58	Super Seniors	YYYYYY	5.58
Sleeping Boba	YYNNYY	3.53			
Alligators and Olives Rule!	YYYYYY	5.58			
Watergators	YNYNNY	2.13			

Problem 11. [11] Let a, b, and c be real numbers such that

$$\frac{2a+b+c}{a} = 3, \quad \frac{a+2b+c}{b} = 4, \quad \frac{a+b+2c}{c} = N.$$

Compute N.

Proposed by Aditya Pahuja

Solution. By subtracting 1 on both sides from each equation, we obtain the following:

$$\begin{aligned} 3-1 &= \frac{a+b+c}{a} \\ 4-1 &= \frac{a+b+c}{b} \\ N-1 &= \frac{a+b+c}{c} \end{aligned}$$

Adding the reciprocals of these expressions together, we get

$$\frac{1}{3-1} + \frac{1}{4-1} + \frac{1}{N-1} = \frac{a}{a+b+c} + \frac{b}{a+b+c} + \frac{c}{a+b+c} = 1.$$

Solving for N gives $\boxed{N=7}$. □

Problem 12. [11] Aditya and Noam start flipping coins at the same time. Aditya flips his coin every 15 seconds, but Noam is eating a banana, so he only flips his coin every 30 seconds. What is the probability that Noam flips heads before (not at the same time as) Aditya?

Proposed by Noam Pasman

> **Remark.** To clear up ambiguity: their first flip is together, at zero seconds.

Solution 1. The probability that Noam flips heads for the first time at the $30n$-second mark and Aditya does not flip heads in the first $30n$ seconds is

$$\left(\frac{1}{2}\right)^{n+1} \cdot \left(\frac{1}{2}\right)^{2n+1} = \left(\frac{1}{2}\right)^{3n+2} = \frac{1}{4} \cdot \left(\frac{1}{8}\right)^{n}$$

since the only way this happens is if Noam flips tails on his first n turns, then flips heads, while Aditya flips tails on his first $2n+1$ turns.

Therefore, the probability that Noam flips heads before Aditya is

$$\sum_{n=0}^{\infty} \frac{1}{4} \cdot \left(\frac{1}{8}\right)^{n} = \frac{1}{4} \cdot \sum_{n=0}^{\infty} \left(\frac{1}{8}\right)^{n} = \frac{1}{4} \cdot \frac{8}{7} = \boxed{\frac{2}{7}}.$$ □

Solution 2. Let p be the answer. Then, $\frac{1}{4}$ of the time, Noam wins on the first flip. Otherwise, he gets to his next flip $\frac{1}{8}$ of the time (if everyone flips tails), at which point his chance of winning is again p. Therefore

$$p = \frac{1}{4} + \frac{p}{8},$$

which gives the answer of $p = \boxed{\frac{2}{7}}$. □

Problem 13. [12] Mr. Kats writes a 2024-digit sequence on the board consisting of zeroes, ones, and twos such that any four consecutive digits sum to a multiple of 3. How many sequences could he have written down?

Proposed by Aditya Pahuja

Solution. Let x_1, x_2, ..., x_{2024} be the sequence that Mr. Kats writes down.

Claim 13.1 — The sequence is uniquely determined by x_1, x_2, and x_3.

Proof. We are given that, for each $1 \le n \le 2021$, $x_n + x_{n+1} + x_{n+2} + x_{n+3}$ is a multiple of 3, so

$$x_{n+3} \equiv -x_n - x_{n+1} - x_{n+2} \pmod 3.$$

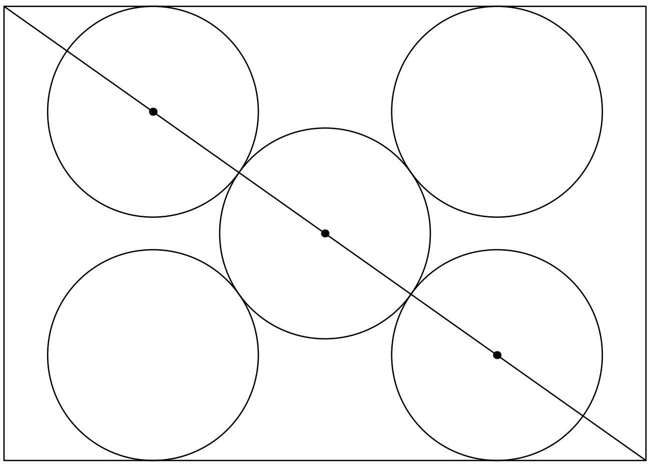
Then, if the values of x_n, x_{n+1}, and x_{n+2} are given, then there is only one possible value of x_{n+3} in $\{0, 1, 2\}$ that satisfies the divisibility relation. Thus, each x_k is determined by the three preceding terms of the sequence, and so x_1, x_2, and x_3 fix the entire sequence. □

From here, there are $3^3 = \boxed{27}$ choices for the first three terms, so we are done. □

Problem 14. [12] In the interior of a cube of side length 6, eight spheres are drawn, each of which is tangent to three of the cube's sides. Then, a ninth sphere is drawn tangent to the first eight spheres. If all nine spheres are congruent, what is their common radius?

Proposed by Aditya Pahuja

Solution. Take a cross section through two opposite edges of the cube. Since the centers of each of the original eight spheres are equidistant to the faces to which they are tangent, we can be sure that the centers of two spheres corresponding to opposite corners will be collinear with the center of the central sphere, as shown in the diagram.



Let r be the common radius of the spheres. We can express the length of the main diagonal in two ways: on one hand, it is equal to $6\sqrt{3}$, but on the other hand, it is equal to the distance between the two far centers, $4r$, plus the distances between each center and the nearest corner, $r\sqrt{3} + r\sqrt{3}$. In total, we end up with

$$r(4 + 2\sqrt{3}) = 6\sqrt{3} \iff r = \frac{6\sqrt{3}}{4 + 2\sqrt{3}} = \frac{3\sqrt{3}}{2 + \sqrt{3}} = 3\sqrt{3} \cdot (2 - \sqrt{3}) = \boxed{6\sqrt{3} - 9}.$$ □

Problem 15. **[13]** Compute the last 3 digits in the decimal representation of $2024^{2025^{2026}}$.

Proposed by Noam Pasman

Solution 1. The last 3 digits of $2024^{2025^{2026}}$ are the same as the last 3 digits of $24^{2025^{2026}}$.

Note that $24^2 = 576 = 600 - 24$. Using this fact, one can repeatedly take powers of 576 and show that

$$24^{10} \equiv 576^5 \equiv 24 \pmod{1000}.$$

This means that $24^a \equiv 24^{10b+a} \pmod{1000}$. Since $2025^{2026} \equiv 5 \pmod{10}$, we have that

$$24^{2025^{2026}} \equiv 24^5 \equiv \boxed{624} \pmod{1000}.$$

□

Solution 2. Obviously, the number is a multiple of 8. We will try to compute the number modulo 125, using the fact that $a^{\varphi(n)} \equiv 1 \pmod{n}$ if $\gcd(a, n) = 1$.

Since $\varphi(125) = 100$, we see that

$$2025^{2026} \equiv 25^{2026} \equiv 25 \pmod{100}.$$

Therefore,

$$2024^{2025^{2026}} \equiv 24^{2025^{2026}} \equiv 24^{25} \pmod{125}.$$

We can then expand 24^{25} as

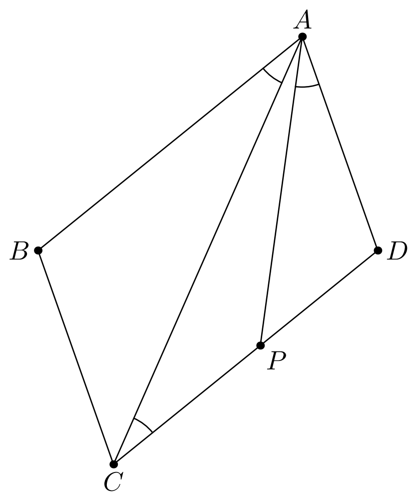
$$(25-1)^{25} = 25^2 \cdot \text{junk} + \binom{25}{1} \cdot 25 \cdot (-1)^{24} + (-1)^{25} \equiv -1 \equiv 124 \pmod{125}.$$

Thus, we need to find the multiple of 8 less than 1000 that is 124 (mod 125), which turns out to be $\boxed{624}$. □

Problem 16. **[13]** In parallelogram $ABCD$, the reflection of diagonal $\overline{AC}$ over the bisector of $\angle BAD$ intersects $\overline{CD}$ at P. If $CD = 9$ and $DP = 4$, compute the length of $\overline{AD}$.

Proposed by Aditya Pahuja

Solution. Notice that $\angle BAC = \angle PAD$ by construction.



This implies that $\angle PAD = \angle BAC = \angle ACD$, so $\triangle DAP \sim \triangle DCA$. Thus,

$$\frac{PD}{AD} = \frac{AD}{CD} \iff AD^2 = PD \cdot CD,$$

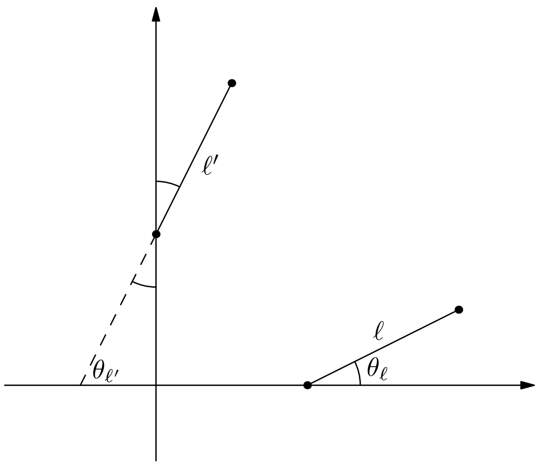
so $AD = \sqrt{4 \cdot 9} = \boxed{6}$. □

Problem 17. [14] Compute
$$\sum_{m=2}^{\infty}\sum_{n=3}^{\infty}\left(\frac{2}{n}\right)^m.$$
Proposed by Aditya Pahuja

Solution. Switching the order of summation, we get
$$\sum_{n=3}^{\infty}\sum_{m=2}^{\infty}\left(\frac{2}{n}\right)^m=\sum_{n=3}^{\infty}\left(\frac{2}{n}\right)^2\sum_{m=0}^{\infty}\left(\frac{2}{n}\right)^m=\sum_{n=3}^{\infty}\frac{4}{n^2}\left(\frac{1}{1-\frac{2}{n}}\right)$$
$$=\sum_{n=3}^{\infty}\frac{4}{n(n-2)}=\sum_{n=3}^{\infty}\left(\frac{2}{n-2}-\frac{2}{n}\right)=\frac{2}{1}+\frac{2}{2}=\boxed{3}.$$
□

Problem 18. [14] Given a segment ℓ in the coordinate plane, we say that θ_ℓ is the smaller of the two angles formed by the line containing ℓ and the x-axis (measured in radians). Compute the sum of θ_ℓ over all segments ℓ whose endpoints have integer coordinates between 0 and 4, inclusive.
Proposed by Aditya Pahuja

Solution. Given a segment ℓ with endpoints satisfying $0 \le x, y \le 4$, let ℓ' be its reflection over the line $y = x$. We see that
$$\theta_\ell+\theta_{\ell'}=\frac{\pi}{2}.$$



Thus,
$$2\sum_{\ell}\theta_\ell=\sum_{\ell}\theta_\ell+\theta_{\ell'}=\binom{25}{2}\cdot\frac{\pi}{2}=150\pi,$$
so the desired sum is half this quantity, namely $\boxed{75\pi}$. □

Problem 19. [15] For how many positive integers n less than 200 do n and $\binom{n}{3}$ have the same last two digits?
Proposed by Aditya Pahuja

Solution. The last two digits of n and $\binom{n}{3}$ are the same if and only if 100 divides
$$\binom{n}{3}-n=\frac{n(n-1)(n-2)}{6}-n=\frac{n^3-3n^2-4n}{6}=\frac{n(n+1)(n-4)}{6}.$$
Equivalently, 600 must divide the numerator of this expression.

Since n, $n+1$, and $n-4$ are pairwise distinct modulo 3, the numerator will always be a multiple of 3, so it suffices to determine when $n(n+1)(n-4)$ is a multiple of 200. From here, we see that

$$n(n+1)(n-4) \equiv 0 \pmod{8}$$

has three solutions (0, 4, and 7) modulo 8 and

$$n(n+1)(n-4) \equiv 0 \pmod{25}$$

has six solutions (0, 4, 9, 14, 19, and 24) modulo 25. Each pair of solutions mod 8 and mod 25 corresponds to a unique solution mod 200 due to Chinese remainder theorem, so we have $3 \cdot 6 = 18$ solutions mod 200. However, there are no positive multiples of 200 less than 200, so we must subtract off $n \equiv 0 \pmod{200}$ as a solution, thereby resulting in an answer of $18 - 1 = \boxed{17}$. □

Problem 20. [Up to 28] Welcome to **USAYNO**!

Instructions: Submit a string of 6 letters corresponding to each statement: put Y if you think the statement is true, N if you think it is false, and X if you do not wish to answer. You will receive $\frac{(n+1)(n+2)}{2}$ *points for n correct answers, but you will receive zero points if any of the questions you choose to answer are incorrect. Note that this means if you submit "XXXXXX" you will get one point.*

(a) If the sum of the divisors of a positive integer n is prime, then n has a prime number of divisors.

(b) In Graphtopia, all flights between pairs of airports go in both directions. From each airport, there are flights to exactly three other airports, and one can travel between any two airports via a sequence of flights. Then, if a random airport is destroyed by a hurricane, it is guaranteed that one can still travel between any pair of intact airports.

(c) There exists an injective function $f\colon \mathbb{R} \to \mathbb{R}$ such that $f(x^2 - 2023x) - f(2x - 2024)^2 \geq \frac{1}{4}$ for all real x.

(d) If a triangle has integer side lengths and integer area, then its area is even.

(e) Given triangle $\triangle ABC$, there is a unique parabola tangent to $\overline{AB}$ at B and $\overline{AC}$ at C.

(f) Let $a_1, a_2, a_3, \ldots$ be a sequence of positive real numbers for which $a_1 + a_2 + a_3 + \cdots$ converges. Suppose that, as n varies, the expression

$$\frac{a_n}{a_{n+1} + a_{n+2} + a_{n+3} + \cdots} = \frac{a_n}{\sum_{k=n+1}^{\infty} a_k}$$

is constant. Then, $a_1, a_2, a_3, \ldots$ is a geometric sequence.

Proposed by Aditya Pahuja, Daniel Potievsky, and Noam Pasman

Solution. The answer is $\boxed{\text{YNNYYY}}$.

(a) Let $n = p_1^{e_1} p_2^{e_2} \cdots p_k^{e_k}$ for distinct primes $p_1, p_2, \ldots, p_k$ and positive integers $e_1, e_2, \ldots, e_k$. Then, the sum of the divisors of n is

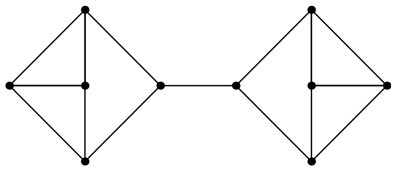
$$\sigma(n) = \prod_{i=1}^{k} \frac{p_i^{e_i+1} - 1}{p_i - 1}.$$

Since each e_i is at least 1, each factor is greater than 1, so $\sigma(n)$ being prime means that the corresponding product has to have only one factor. In other words, n must be a prime power, say p^e. If $e+1$ has a divisor $1 < d < e+1$, then we can extract a factor of

$$1 < \frac{p^d - 1}{p - 1} < \sigma(n),$$

so $\sigma(n)$ would not be prime in this case. Thus, $\sigma(n)$ being prime implies that $e+1$, the number of divisors of n, is prime.

(b) The smallest counterexample is shown below. To construct it, note that there must be a central airport (the one destroyed by the hurricane) that is connected to two independent branches of Graphtopia. Trying to connect the airports within each branch while making sure each airport has exactly three flights to other airports will eventually lead to one of many possible route maps.



> **Remark.** For those knowledgeable on graph theory terminology, the problem statement can be stated more concisely as follows:
>
> All 3-regular connected graphs are 2-connected.

> **Remark.** It is an interesting problem to consider for which integers there is a counterexample with exactly that many airports.

(c) Setting $x = 1$ gives

$$f(-2022) - f(-2022)^2 \geq \frac{1}{4} \iff 0 \geq \left(f(-2022) - \frac{1}{2}\right)^2,$$

so $f(-2022) = \frac{1}{2}$ (otherwise the right-hand side would be positive). Similarly, via $x = 2024$ we get

$$0 \geq \left(f(2024) - \frac{1}{2}\right)^2$$

implying $f(2024) = \frac{1}{2}$. Thus, $f(-2022) = f(2024)$, which means f can't be injective.

(d) Scale down the triangle so that the new triangle has integer side lengths a, b, and c satisfying $\gcd(a, b, c) = 1$ and rational area K. It then suffices to show that K is an even integer.

By Heron's formula,

$$K^2 = \frac{a+b+c}{2} \cdot \frac{-a+b+c}{2} \cdot \frac{a-b+c}{2} \cdot \frac{a+b-c}{2}.$$

We know that $4K = \sqrt{16K^2}$ is rational and the square root of an integer, so it must be an integer. Thus, K can be expressed as $\frac{L}{4}$ for some integer L. In order for K to be an even integer, we need $8 \mid L$, which necessitates

$$64 \mid L^2 = 16K^2 = (a+b+c)(-a+b+c)(a-b+c)(a+b-c).$$

If exactly one of the side lengths (WLOG, a) is even, then we can pair terms to get

$$16K^2 = (a+b+c)(a+b-c) \cdot (-a+b+c)(a-b+c) = ((a+b)^2 - c^2)(c^2 - (a-b)^2).$$

Squares of odd numbers are always $1 \pmod 8$, so both factors here are $1 - 1 \equiv 0 \pmod 8$, hence the overall product is divisible by 64.

On the other hand, if either one or three of the numbers are odd, then say WLOG c is odd; this implies $a \pm b$ is even. Again writing

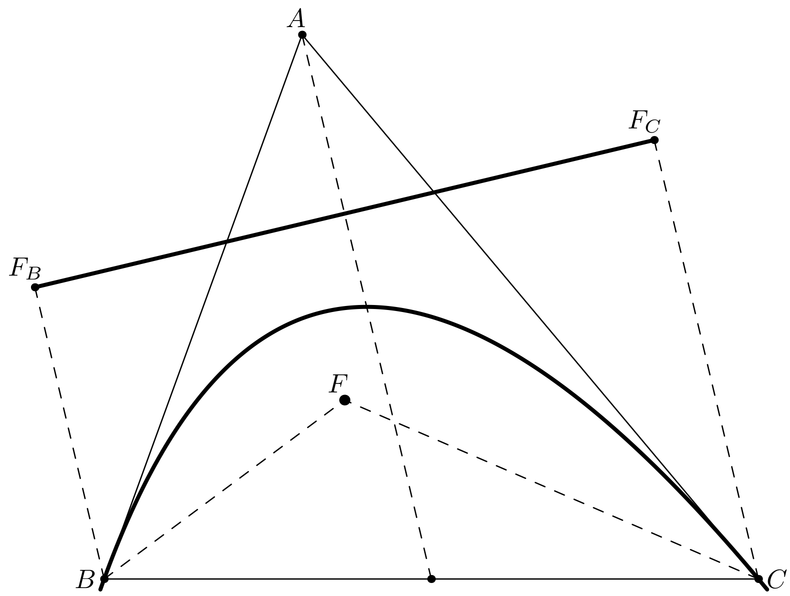
$$16K^2 = (a+b+c)(a+b-c) \cdot (-a+b+c)(a-b+c) = ((a+b)^2 - c^2)(c^2 - (a-b)^2),$$

we see that the product is $(4K)^2 \equiv -c^4 \equiv -1 \pmod 4$. However, there is no integer n for which $n^2 \equiv -1 \pmod 4$, which contradicts the fact that $4K$ is an integer. Therefore, this case is impossible.

(e) We will show that $\triangle ABC$ uniquely determines a parabola by showing that there is only one possible location of the focus; this is enough because we can construct the directrix by using the reflective property of the parabola (namely that the focus, when reflected over a tangent line to the parabola, will end up on its directrix).

Suppose F is the focus of a parabola with the desired tangencies. Then, reflect F over $\overline{AB}$ to get F_B and over $\overline{AC}$ to get F_C. We know that line $\overline{F_BF_C}$ is the directrix of the parabola, so $\overline{BF_B}$ and $\overline{CF_C}$ are perpendicular to $\overline{F_BF_C}$ (since both B and C are on the parabola), so the two lines are parallel. This also tells us that

$$\angle F_BBA + \angle F_CCA = \angle A.$$



If we place the parabola in the coordinate plane so that the axis of symmetry is vertical (parallel to the y-axis) then it is well-known that the x-coordinate of A is the average of the x-coordinates of B and C (you can prove this by just coordinate bashing), meaning the vertical line through A bisects $\overline{BC}$ — in other words, the A-median of $\triangle ABC$ is perpendicular to the directrix.

This is enough information to recover the focus only using $\triangle ABC$: specifically, the focus is the intersection of the reflections of the lines through B and C parallel to the A-median over $\overline{AB}$ and $\overline{AC}$ respectively.

> **Remark.** It turns out that F is the unique point such that $\triangle BFA \sim \triangle AFC$, known as the **$A$-Dumpty point** of $\triangle ABC$.

(f) This solution relies on the following fact:

Theorem 20.1 (Baseball theorem)**.** For any complex numbers a, b, c, d with b and d nonzero,

$$\frac{a}{b} = \frac{c}{d} = r \implies \frac{a-c}{b-d} = r.$$

Proof. Write $a = br$ and $c = dr$, so that

$$\frac{a-c}{b-d} = \frac{rb-rd}{b-d} = r.$$

This of course generalizes to any linear combination of the numerator and denominator. □

We know that there is a constant c such that

$$c = F(n) = \frac{a_n}{a_{n+1} + a_{n+2} + a_{n+3} + \cdots}.$$

Applying baseball theorem on $F(n)$ and $F(n+1)$ yields

$$c = \frac{a_n - a_{n+1}}{(a_{n+1} + a_{n+2} + a_{n+3} + \cdots) - (a_{n+2} + a_{n+3} + \cdots)} = \frac{a_n - a_{n+1}}{a_{n+1}} = \frac{a_n}{a_{n+1}} - 1,$$

so $\frac{a_n}{a_{n+1}} = c + 1$ is constant, meaning $a_1, a_2, a_3, \ldots$ is geometric.

$\square$

> **Remark.** This problem was inspired by the following classic problem:
> An ant is at 0 on the number line. Each minute, it randomly picks a positive integer, with n being chosen with probability $\frac{1}{2^n}$, and jumps forward n steps. What is the probability that it eventually lands on 2024?

Problem 21. [16] Given a prime p, we say that positive integer n is pseudo-cyclic with signature p if n, along with all the numbers formed by cyclically permuting the digits of n (preserving leading zeroes), are multiples of p. For example, 1034 is pseudo-cyclic with signature 11 because 1034, 0341, 3410, and 4103 are all multiples of 11.

If n is a six-digit pseudo-cyclic number, compute the sum of all possible values of its signature.

Proposed by Aditya Pahuja

Solution. Let $n = 10^5 a_5 + 10^4 a_4 + 10^3 a_3 + 10^2 a_2 + 10a_1 + a_0$, where the a_i are digits. Then, we are given that p divides

$$X = 10^5 a_{k+5} + 10^4 a_{k+4} + 10^3 a_{k+3} + 10^2 a_{k+2} + 10a_{k+1} + a_k,$$
$$Y = 10^5 a_k + 10^4 a_{k+5} + 10^3 a_{k+4} + 10^2 a_{k+3} + 10a_{k+2} + a_{k+1}.$$

for each $0 \le k \le 5$, with indices taken modulo 6. Thus,

$$p \mid 10Y - X = (10^6 - 1)a_k,$$

so either p divides $10^6 - 1$ or p divides each digit of n. In the first case, p is a prime divisor of

$$10^6 - 1 = (10-1)(10+1)(10^2 - 10 + 1)(10^2 + 10 + 1) = 9 \cdot 11 \cdot 91 \cdot 111 = 3^3 \cdot 7 \cdot 11 \cdot 13 \cdot 37.$$

In the second case, p can be any single-digit prime. Each of these values is achievable, in the first case by just taking $n = 10^6 - 1$ and in the second case by taking $a_k = p$ for all k.

Thus, the answer is

$$2 + 3 + 5 + 7 + 11 + 13 + 37 = \boxed{78}.$$

$\square$

Problem 22. [16] If x, y, z are positive real numbers such that $xyz(x+y+z) = 2024$, find the smallest possible value of $(x+y)(y+z)$.

Proposed by Daniel Potievsky

Solution 1. Let $\triangle ABC$ be a triangle with $AB = c = x + y$, $BC = a = y + z$, and $CA = b = z + x$. Then, letting $s = \frac{a+b+c}{2} = x + y + z$, the area of $\triangle ABC$ is equal to

$$\sqrt{s(s-a)(s-b)(s-c)} = \sqrt{(x+y+z)xyz} = \sqrt{2024} = 2\sqrt{506}.$$

However, it is also equal to

$$\frac{1}{2}ab\sin\angle C = \frac{1}{2}(x+y)(y+z)\sin C = 2\sqrt{506},$$

which, because $\sin C \le 1$, implies that

$$(x+y)(y+z) = \frac{2 \cdot 2\sqrt{506}}{\sin \angle C} \ge \boxed{4\sqrt{506}},$$

with equality when $\angle C = 90°$. □

Solution 2. We can write

$$(x+y)(y+z) = xz + y(x+y+z) = xz + \frac{2024y}{xyz} = zx + \frac{2024}{xz}.$$

By the AM-GM inequality,

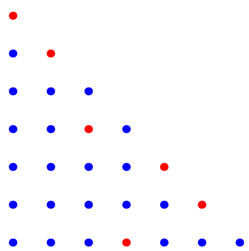
$$zx + \frac{2024}{xz} \ge 2 \cdot \sqrt{zx \cdot \frac{2024}{xz}} = 2\sqrt{2024},$$

so $(x+y)(y+z)$ is at least $\boxed{4\sqrt{506}}$. Equality is achieved if and only if $xz = \sqrt{2024}$; checking that there exist positive reals (x, y, z) satisfying these constraints is routine. □

Problem 23. **[17]** Let S_n be the set of points in the coordinate plane with nonnegative integer coordinates summing to at most n. For integers $n \ge 1$, let a_n be the number of ways to choose n points in S_n such that no two share an x- or a y-coordinate. What is $a_1 + a_2 + \cdots + a_{10}$?

Proposed by Aditya Pahuja

Solution. Color the chosen points red and the rest of the points blue. Observe that S_n is a triangular grid with $n+1$ rows and columns; here is one valid coloring of S_6 pictured below.



The key observation, then, is that each valid coloring has an empty row. Say the kth row from the bottom is empty. Then, the $n-k$ rows above have $n-k$ red points, which forces all of those red points to be along the main diagonal. On the other hand, the points below the empty row can be chosen top-down: the $(k-1)$th row has two spaces for red points, and each row below will also be left two valid spaces for a red point — this leaves us with 2^{k-1} ways to pick the red points. Therefore,

$$a_n = 2^n + 2^{n-1} + \cdots + 2^1 + 2^0 = 2^{n+1} - 1,$$

so

$$a_1 + a_2 + \cdots + a_{10} = 2^2 + 2^3 + \cdots + 2^{11} - 10 = 2^{12} - 2^2 - 10 = 4096 - 14 = \boxed{4082}.$$ □

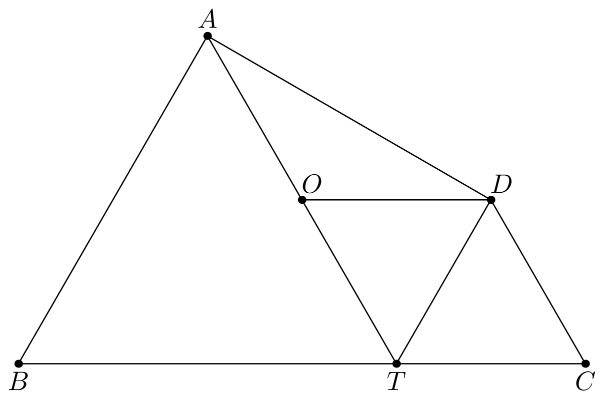
Problem 24. **[17]** In convex quadrilateral $ABCD$, the lengths of sides $\overline{AB}$, $\overline{BC}$, and $\overline{CD}$ are 4, 6, and 2 respectively, and the measure of $\angle ABC$ is $60°$. Let T be the point such that $BT = 4$ and $CT = 2$. When the circumradius of $\triangle ADT$ is minimal across all possible choices of D, what is the area of $ABCD$?

Proposed by Aditya Pahuja

Solution. By the law of sines, the circumradius of $\triangle ADT$ is

$$\frac{AT}{2\sin\angle ADT} = \frac{4}{2\sin\angle ADT}.$$

Therefore, to minimize this quantity, we want to maximize $\sin \angle ADT$, which happens when $\angle ADT = 90°$.



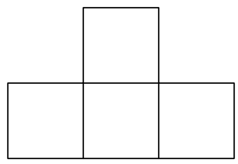
Letting O be the circumcenter of $\triangle ADT$ (which is the midpoint of $\overline{AT}$), we have

$$OD = DC = CT = TO = 2,$$

which means $ODCT$ is a rhombus. Thus, $\angle OTC = 120^\circ$ implies $\angle TCD = 60^\circ$, so $\triangle DCT$ is equilateral. This also gives us $\angle ATD = 60^\circ$, so $AD = DT\sqrt{3} = 2\sqrt{3}$. The answer is then

$$[ABT] + [ATD] + [CTD] = \frac{4^2\sqrt{3}}{4} + \frac{2 \cdot 2\sqrt{3}}{2} + \frac{2^2\sqrt{3}}{4} = 4\sqrt{3} + 2\sqrt{3} + \sqrt{3} = \boxed{7\sqrt{3}}. \qquad \square$$

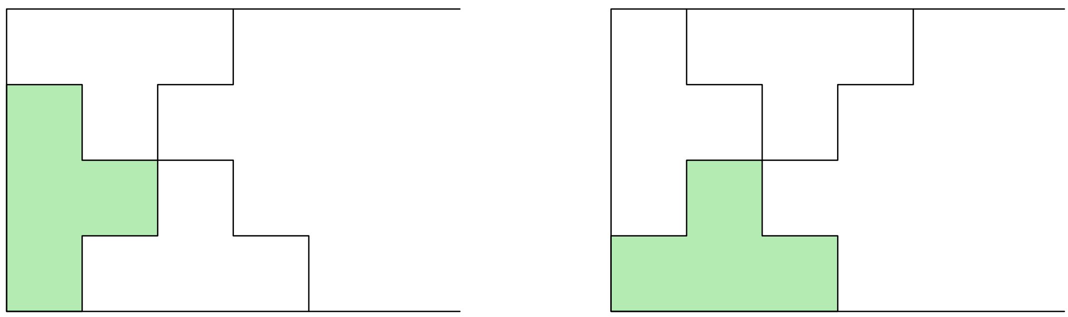
Problem 25. [18] Let N be the number of ways to tile a 4×2024 board using only T-tetrominoes (pictured below). Find the sum of the (not necessarily distinct) primes in the prime factorization of N. For example, if $N = 12$, submit $2 + 2 + 3 = 7$.



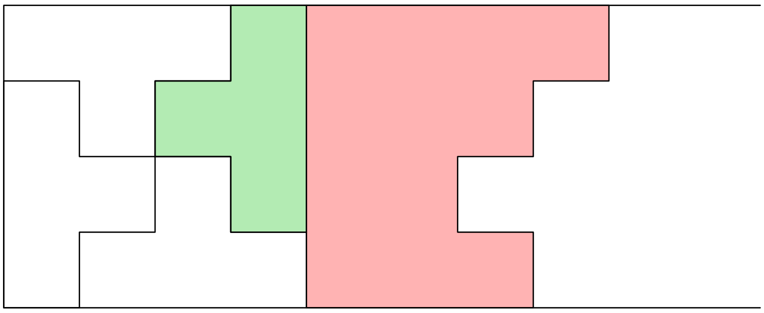
Proposed by Noam Pasman

Solution 1. We will refer to orientations of T-tetrominoes depending on where the cell sticking out is relative to the line of three cells. For example, the T-tetromino shown in the problem statement is in the up orientation.

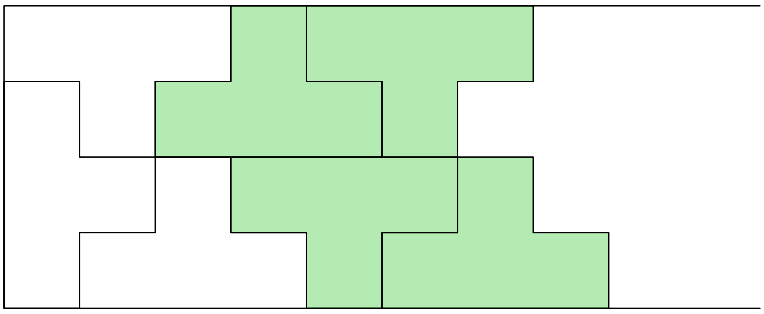
To fill the bottom left corner of the board, we must either have an up tetromino or a right tetromino. Each of these then forces the orientations of two more pieces in the board as shown below:



Notice that these two are simply mirror images of each other, so we'll just work with the first one and multiply by 2. In order to fill the second cell in the third column, we must either have a left or an up tetromino. In the first case, we have filled a complete 4 by 4 section of the board, so to fill the next 4 columns we will eventually end up with either the same setup as we just had or its reflection.

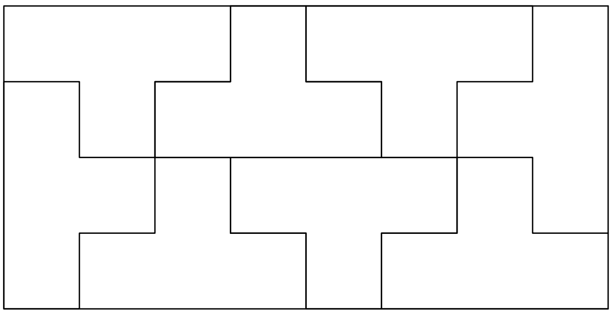


Meanwhile, in the second case we are forced to place three more tetrominoes as shown below, and we end up with the same situation for the next 4 columns.



This means that in every block of 4 columns, we have 3 choices for how to finish filling it in (except for the last one, where we only have one choice). Since there are $\frac{2024}{4} - 1 = 505$ of these blocks, we have $2 \cdot 3^{505}$ ways to tile the entire board, where we multiply by 2 for whether the bottom left corner has an up or a right tetromino. Thus, the answer is $2 + 3 \cdot 505 = \boxed{1517}$. $\square$

Solution 2. Call a tiling of a $4 \times n$ grid *rigid* if no $4 \times m$ grid within the $4 \times n$ grid is tiled by the tetrominoes. For example, the following picture is a rigid tiling with $n = 8$:



We can see that, if n is a multiple of 4, then it admits two rigid tilings, and otherwise it admits zero rigid tilings. This means, for each sequence of positive integers $a_1, a_2, \ldots, a_k$ satisfying $4a_1 + 4a_2 + \cdots + 4a_k = 2024$, there are 2^k corresponding tilings, as we can interpret $(4a_1, 4a_2, \ldots, 4a_k)$ as the tuple of lengths of the rigid subgrids of some tiling. For a given k, there are $\binom{505}{k-1}$ choices of $(a_1, a_2, \ldots, a_k)$, so the total number of tilings is

$$\sum_{k=1}^{506} 2^k \binom{505}{k-1} = 2 \sum_{k=0}^{505} 2^k \binom{505}{k} = 2 \cdot (2+1)^{505} = 2 \cdot 3^{505}.$$

Thus, the answer is again $2 + 3 \cdot 505 = \boxed{1517}$. $\square$

Problem 26. [18] Each face of a cube is labeled with a nonnegative integer. Then, each vertex of the cube is assigned the product of the numbers on the three faces containing that vertex. In how many possible ways can we choose the face labels so that the sum of the vertex labels is 81, where rotations and reflections of labelings are considered distinct?

Proposed by Aditya Pahuja

Solution. Let a, b, c, d, e, f be the numbers on the faces, with a being opposite d, b being opposite e, and c being opposite f on the cube. Then, the sum of the vertex labels is

$$81 = abc + ace + aef + afb + dbc + dce + def + dfb = (a+d)(b+e)(c+f).$$

Each of $a+d$, $b+e$, and $c+f$ must be a power of 3, say 3^x, 3^y, and 3^z respectively for nonnegative integers x, y, and z. For each choice of (x, y, z), there are 3^x+1 ways to choose (a, d), 3^y+1 ways to choose (b, e), and 3^z+1 ways to choose (c, f). Therefore, each (x, y, z) yields $(3^x+1)(3^y+1)(3^z+1)$ valid labelings. Our task, then, is to compute

$$\sum_{x+y+z=4} (3^x+1)(3^y+1)(3^z+1)$$

where the sum is over all triples of nonnegative integers summing to 4. Noting that there are $\binom{4+2}{2} = 15$ such triples, the sum can be simplified to

$$\sum_{x+y+z=4} (3^4 + 3^{4-x} + 3^{4-y} + 3^{4-z} + 3^x + 3^y + 3^z + 1) = 15(81+1) + 3 \cdot \sum_{x+y+z=4} \left(3^x + 3^{4-x}\right).$$

We can then see that the number of triples with $x = k$ is always $5-k$ and the number of triples with $x = 4-k$ is always $k+1$, so 3^k appears in $(5-k)+(k+1) = 6$ addends of the remaining sum. Thus, we get

$$15 \cdot 82 + 3 \sum_{x+y+z=4} \left(3^x + 3^{4-x}\right) = 15 \cdot 82 + 3 \cdot 6 \cdot \sum_{k=0}^{4} 3^x = 15 \cdot 82 + 18 \cdot 121 = \boxed{3408}. \qquad \square$$

Problem 27. [19] A regular 2024-gon $A_1A_2 \cdots A_{2024}$ is inscribed in the unit circle in the complex plane. Given that A_1, A_2, $\ldots$, A_{1012} have nonnegative imaginary parts, the difference between the largest and smallest possible value of the sum of their imaginary parts can be expressed as $\tan\theta$, with $0 < \theta < \frac{\pi}{2}$. Compute θ.

Proposed by Aditya Pahuja

Solution 1. Let $\alpha = \frac{2\pi}{2024}$ and $0 \le x < \alpha$. Then, the sum of the imaginary parts of A_1, A_2, $\ldots$, A_{1012} is

$$\begin{aligned}\sum_{k=0}^{1011} \sin(k\alpha + x) &= \sum_{k=0}^{1011} \sin(k\alpha)\cos(x) + \sin(x)\cos(k\alpha)\\ &= \cos(x)\sum_{k=0}^{1011} \sin(k\alpha) + \sin(x)\sum_{k=0}^{1011}\cos(k\alpha).\end{aligned}$$

Note that the nonnegativeness condition is satisfied because $k\alpha + x$ is forced to be between 0 and π here. The second sum is equal to $\sin(x)$ because, by pairing $\cos(k\alpha)$ with $\cos((1012-k)\alpha)$, we get a bunch of sums that add to zero, leaving us with $\cos(0) = 1$. Letting S be the first sum (without the $\cos(x)$ attached), we see that

$$\begin{aligned}S\sin(\alpha) &= \sum_{k=0}^{1011} \sin(k\alpha)\sin(\alpha)\\ &= \frac{1}{2}\sum_{k=0}^{1011} \cos((k-1)\alpha) - \cos((k+1)\alpha)\\ &= \frac{1}{2}(\cos(-\alpha) + \cos(0) - \cos(1011\alpha) - \cos(1012\alpha))\\ &= 1 + \cos(\alpha)\end{aligned}$$

which, using double angle formulas, tells us that

$$S = \frac{1+\cos(\alpha)}{\sin(\alpha)} = \frac{2\cos^2(\frac{\alpha}{2})}{2\sin(\frac{\alpha}{2})\cos(\frac{\alpha}{2})} = \cot\left(\frac{\alpha}{2}\right).$$

Therefore, we want the maximum and minimum of the expression

$$\cos(x)\cot\left(\frac{\alpha}{2}\right) + \sin(x) = \left(\cos(x)\cdot\frac{\cot(\frac{\alpha}{2})}{\sqrt{1+\cot^2(\frac{\alpha}{2})}} + \sin(x)\cdot\frac{1}{\sqrt{1+\cot^2(\frac{\alpha}{2})}}\right)\cdot\sqrt{1+\cot^2\left(\frac{\alpha}{2}\right)},$$

which, since $1+\cot^2 y = \csc^2 y$, is equal to

$$\csc\left(\frac{\alpha}{2}\right)\left(\cos(x)\cos\left(\frac{\alpha}{2}\right) + \sin(x)\sin\left(\frac{\alpha}{2}\right)\right) = \csc\left(\frac{\alpha}{2}\right)\cos\left(x-\frac{\alpha}{2}\right).$$

Recalling that $0 \le x < \alpha$, this expression is minimized when x is as far away from $\frac{\alpha}{2}$ as possible, i.e. at $x = 0$, and it is maximized when $\cos(x-\frac{\alpha}{2}) = 1$, i.e. at $x = \frac{\alpha}{2}$. The difference between the two extrema is

$$\csc\left(\frac{\alpha}{2}\right) - \cot\left(\frac{\alpha}{2}\right) = \frac{1-\cos(\frac{\alpha}{2})}{\sin(\frac{\alpha}{2})} = \frac{2\sin^2(\frac{\alpha}{4})}{2\sin(\frac{\alpha}{4})\cos(\frac{\alpha}{4})} = \tan\left(\frac{\alpha}{4}\right),$$

which means that the answer is $\frac{\alpha}{4} = \boxed{\frac{\pi}{4048}}$. □

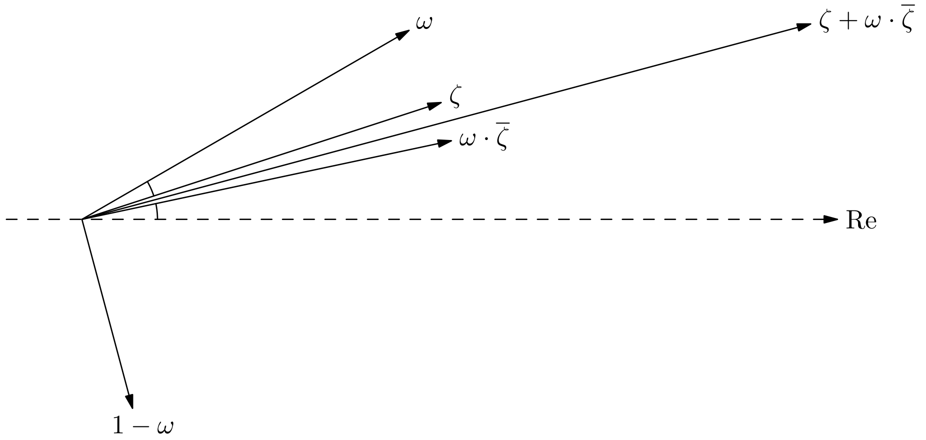
Solution 2. Alternatively, you can set up the calculation using complex numbers. Let $\omega = \cos(\alpha) + i\sin(\alpha)$ and let $\zeta = \cos(x) + i\sin(x)$, with α and x retaining their meanings from the previous solution. Then, the sum of the imaginary parts is given by

$$\sum_{k=0}^{1011} \frac{\omega^k\cdot\zeta - \overline{\omega}^k\cdot\overline{\zeta}}{2i}.$$

Using geometric series and the fact that $\omega^{1012} = -1$, this evaluates to

$$\frac{1}{2i}\cdot\left(\frac{\zeta\cdot\omega^{1012}-\zeta}{\omega-1} - \frac{\overline{\zeta}\cdot\overline{\omega}^{1012}-\overline{\zeta}}{\overline{\omega}-1}\right) = \frac{1}{2i}\cdot\left(\frac{-2\zeta}{\omega-1} - \frac{2\overline{\zeta}\cdot\omega}{\omega-1}\right) = \frac{\zeta+\omega\cdot\overline{\zeta}}{i(1-\omega)}.$$

Geometrically, we have the following setup:



We already know that the desired expression is a positive real number because it is a sum of a bunch of positive real numbers, so it suffices to minimize and maximize its magnitude. Notably, since 0, ζ, $\omega\cdot\overline{\zeta}$, and $\zeta+\omega\cdot\overline{\zeta}$ form a rhombus, the magnitude of $\zeta+\omega\cdot\overline{\zeta}$ is twice the length of the altitude from 0 to the line through ζ and $\omega\cdot\overline{\zeta}$, i.e.

$$2\cos\left(\frac{\alpha-2x}{2}\right)$$

because the angle between the two vectors is $|\alpha - 2x|$. Also, $1-\omega$ has a magnitude of $2\sin(\frac{\alpha}{2})$, so

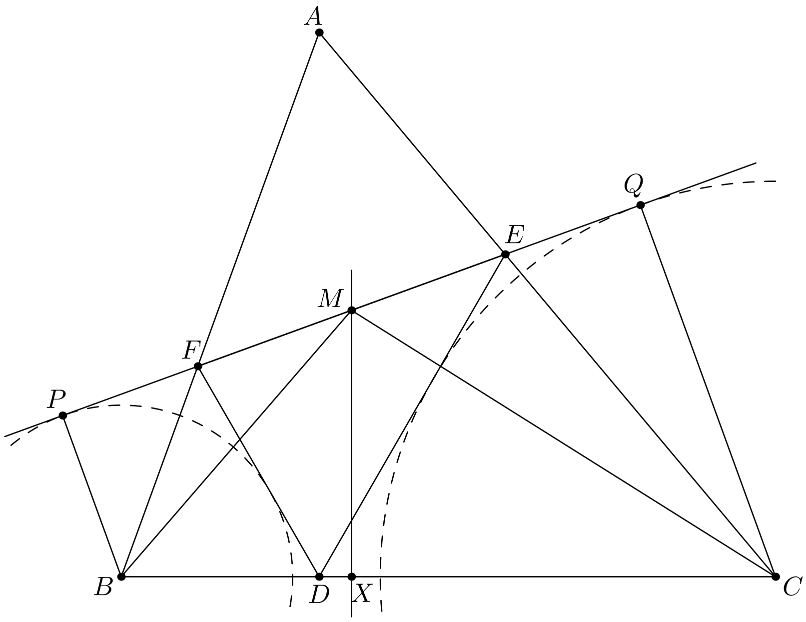
$$\left|\frac{\zeta + \omega \cdot \overline{\zeta}}{i(1-\omega)}\right| = \frac{2\cos(\frac{\alpha}{2} - x)}{2\sin(\frac{\alpha}{2})},$$

which is minimized at $x = 0$ and maximized at $x = \frac{\alpha}{2}$. From here, we see that the minimum is $\cot(\frac{\alpha}{2})$ and the maximum is $\csc(\frac{\alpha}{2})$, and we finish as in the previous solution. □

Problem 28. **[19]** In acute triangle $\triangle ABC$, altitudes $\overline{AD}$, $\overline{BE}$, and $\overline{CF}$ are drawn. Point X is on $\overline{BC}$ such that the line through X perpendicular to $\overline{BC}$ bisects $\overline{EF}$. Given that the distance from B to $\overline{DF}$ is 20, the distance from C to $\overline{DE}$ is 24, and the length of $\overline{BC}$ is 2024, compute $CX - BX$.

Proposed by Aditya Pahuja

Solution 1. Let ω_B be the circle centered at B tangent to $\overline{DF}$ and let ω_C be the circle centered at C tangent to $\overline{DE}$.



Since quadrilaterals $ABDE$ and $ACDF$ are cyclic,

$$\angle FDB = \angle FAC = \angle BAC = \angle BAE = \angle EDC,$$

so line $\overline{BC}$ is the external bisector of $\angle FDE$. Similarly, $\overline{AB}$ is the external bisector of $\angle DFE$ and $\overline{AC}$ is the external bisector of $\angle DEF$, which means that ω_B is the E-excircle and ω_C is the F-excircle in $\triangle DEF$. In particular, line $\overline{EF}$ is tangent to the two circles, say at P and Q respectively.

Then, it is a well-known property of excircles that the lengths of $\overline{EP}$ and $\overline{FQ}$ are both half the perimeter of $\triangle DEF$. This implies $FP = EQ$, so the midpoint M of $\overline{EF}$ is also the midpoint of $\overline{PQ}$.

At this point, we can say that

$$BM^2 - BP^2 = PM^2 = QM^2 = CM^2 - CQ^2$$

by the Pythagorean theorem, so

$$BX^2 - BP^2 = BM^2 - MX^2 - BP^2 = CM^2 - MX^2 - CQ^2 = CX^2 - CQ^2.$$

Thus,

$$CX^2 - BX^2 = CQ^2 - BP^2 = 24^2 - 20^2$$

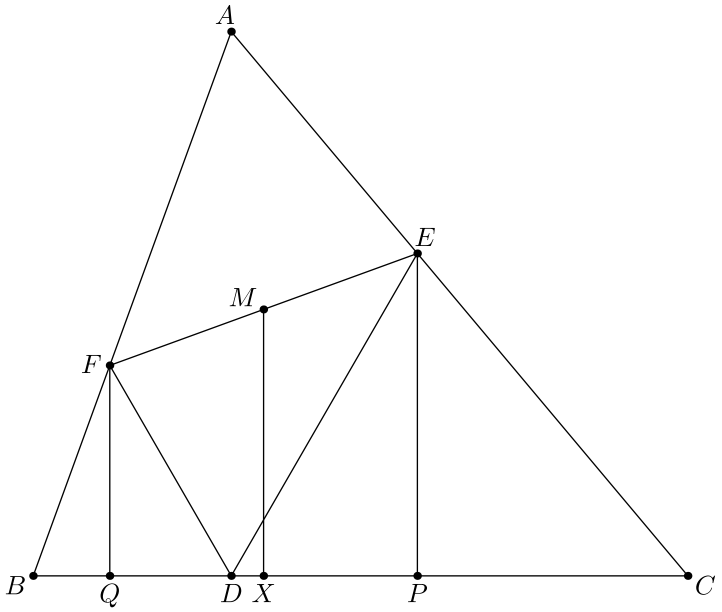
so the final answer is

$$CX - BX = \frac{24^2 - 20^2}{CX + BX} = \frac{176}{2024} = \boxed{\frac{2}{23}}.$$

□

Remark. The final Pythagorean theorem computation is really just a radical axis argument in disguise: since M has equal powers with respect to ω_B and ω_C and line $\overline{MX}$ is perpendicular to the line through their centers, $\overline{MX}$ is the radical axis of the circles. We then use the fact that X has equal power with respect to the circles to extract the equation above.

Solution 2. Let P and Q be the feet of E and F on $\overline{BC}$ respectively.



Since $FM = ME$, we have $QX = QP$ so that $CX - BX = CP - BQ$. We then see that $BF = BC \cos B$ and $CE = BC \cos C$. This allows us to express the distance from B to $\overline{DF}$ as $BF \sin C$ because $\angle BFD = \angle C$ (due to $ACDF$ being cyclic); therefore,

$$20 = BF \sin C = BC \cos B \sin C = 2024 \cos B \sin C.$$

Analogously,

$$24 = CE \sin B = BC \cos C \sin B = 2024 \cos C \sin B.$$

To finish, the desired quantity is

$$\begin{aligned} CP - BQ &= CE \cos C - BF \cos B = BC(\cos^2 C - \cos^2 B) \\ &= 2024(\cos C + \cos B)(\cos C - \cos B) \\ &= 2024 \left(2 \cos\left(\frac{B+C}{2}\right) \cos\left(\frac{B-C}{2}\right)\right) \left(2 \sin\left(\frac{B+C}{2}\right) \sin\left(\frac{B-C}{2}\right)\right) \\ &= 2024 \sin(B+C) \sin(B-C) \\ &= 2024(\sin B \cos C + \sin C \cos B)(\sin B \cos C - \sin C \cos B) \\ &= 2024 \cdot \frac{24+20}{2024} \cdot \frac{24-20}{2024} = \frac{44 \cdot 4}{2024} \\ &= \boxed{\frac{2}{23}}. \end{aligned}$$

□

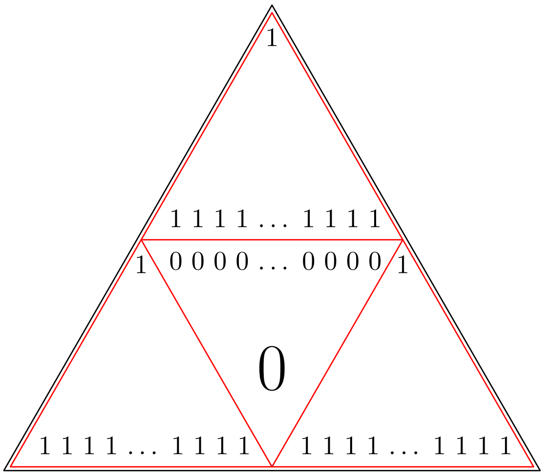
Problem 29. [20] How many pairs of nonnegative integers (n, k) are there such that $n \geq k$, $n + k < 64$, and $\binom{n}{k}$ is odd?

Proposed by Aditya Pahuja

Solution 1. Consider the first 64 rows of Pascal's triangle, reduced modulo 2, drawn so that the numbers in the triangle form an equilateral triangle lattice.

Claim 29.1 — This triangle is symmetric under the rotations and reflections that map the triangle onto itself.

Proof. We can show by induction that the 2^mth row of the triangle always contains only ones, since a triangle of size 2^{m+1} has three copies of the 2^m-size triangle at its corners, and all copies are oriented the same way. This shows us that the boundary of our triangle is all ones. Note also that the central quarter-triangle can only ever contain zeroes.



Then, let a, b, c be numbers in the triangle such that a and b are adjacent in some row, and c is in the row below a, b and adjacent to both:

$$\begin{matrix} a & & b \\ & c & \end{matrix}$$

Pascal's identity tells us that $a + b = c$, and so $a + b - c \equiv a + b + c \equiv 0 \pmod 2$. We can extract the three congruences

$$\begin{aligned} a + b &\equiv c \pmod 2 \\ b + c &\equiv a \pmod 2 \\ c + a &\equiv b \pmod 2. \end{aligned}$$

If we fill in Pascal's triangle normally (top-down), the relation above tells us that this will yield the same result as if we rotated the triangle so one of the other vertices was on top and then filled it in top-down according to the congruence rule. This establishes rotational symmetry modulo 2. Using the fact that $\binom{n}{k} = \binom{n}{n-k}$, we already have reflective symmetry in one way, and then rotation gives us the other two reflective symmetries. □

Claim 29.2 — The line $n + k = 63$ contains exactly one odd number.

Proof. Rotating by 120° sends this line to the line $n = 2k$, which contains the odd number $\binom{0}{0}$; the rest are even because

$$\binom{2n}{n} = \binom{2n-1}{n-1} + \binom{2n-1}{n} = 2\binom{2n-1}{n-1}.$$

Due to the symmetry established in Claim 29.1, $\binom{63}{0}$ must therefore be the only odd $\binom{n}{k}$ with $n + k = 63$. □

Claim 29.3 — The triangle contains $3^6 = 729$ odd numbers.

Proof. This follows from induction: in general, the first 2^m rows contain 3^m odd numbers. The base case $m = 0$ is obvious because $\binom{0}{0} = 1$. Referring to the figure from Claim 29.1, if we assume for inductive hypothesis that each of the corner 2^{m-1}-size triangles has 3^{m-1} odd numbers (ones), then we have a total of $3 \cdot 3^{m-1}$ odd numbers among the first 2^m rows as desired. $\square$

Remark. When first 2^m rows of Pascal's triangle are reduced modulo 2, the ones form a shape that looks like Sierpinski's fractal as m increases.

Therefore, the set of numbers above the line $n+k = 63$ contains the same number T of odd numbers as the set of numbers below that line, so $2T+1 = 3^6$. Thus, the answer is $T+1 = \frac{3^6+1}{2} = \boxed{365}$. $\square$

Solution 2. We will characterize all (n, k) pairs for which $\binom{n}{k}$ is odd. The relevant fact here is the following:

Theorem 29.1. The largest integer m such that $2^m \mid n!$ is $n - s(n)$, where $s(n)$ is the sum of the digits in the binary representation of n.

Proof. The idea is to consider how $s(n)$ changes as n is incremented.

Suppose the binary representation of n has a string of d ones at the end (possibly $d = 0$, if n is even). When 1 is added to n, we get

$$\underbrace{A0111\dots1}_{n} + 1 = \underbrace{A1000\dots0}_{n+1}$$

where A is some binary string. The d ones at the end all become zeroes upon incrementing, and a single new one is produced. Thus, $s(n+1) = s(n) - d + 1$. Notably, d turns out to be the largest integer such that $2^d \mid n+1$.

Now, we can prove the theorem via induction. The base case $n = 0$ is obvious, so suppose the theorem holds for some n. Then, the number of twos dividing $(n+1)!$ is

$$(n - s(n)) + d = (n - s(n)) + (s(n) + 1 - s(n+1)) = (n+1) - s(n+1)$$

where d has the same definition as earlier, so we are done. $\square$

Now, in order for $\binom{n}{k} = \frac{n!}{k!(n-k)!}$ to be odd, we need

$$(n - s(n)) - (k - s(k)) - ((n-k) - s(n-k)) = 0.$$

This is equivalent to saying $s(n) = s(k) + s(n-k)$, a statement which can only hold if the addition $k + (n-k)$ has no carrying when done in binary. This can be rephrased as saying a digit of k can be one only if the corresponding digit in n is also one.

Replacing 64 with an arbitrary power of two 2^m in the original problem, let a_m be the answer. We will construct a recursion for a_m, noting that $a_1 = 2$. We see that n and k have at most m digits in their binary representation since both have to be less than 2^m, so we will do casework on the first digit of n's binary representation. In the first case,

$$\begin{aligned} n &= 1\_\_\_ \cdots \_\_ \\ k &= 0\_\_\_ \cdots \_\_ \end{aligned}$$

where the $m-1$ blank spaces will be some binary digits. Since $n+k$ is already at least 2^{m-1}, we need the unfilled portions ($n - 2^{m-1}$ and k) to sum to less than 2^{m-1}. Moreover, as shown above a digit of k can be one only if the corresponding digit in n is also one — this rule is independent of the leading digits, and so the number of ways to fill in the blank spaces has to be just a_{m-1}.

The other case is the one where both numbers are less than 2^{m-1} and so we have no restrictions on their size: in this case, we still have

$$\begin{aligned} n &= \_\,\_\,\_ \cdots \_\,\_ \\ k &= \_\,\_\,\_ \cdots \_\,\_ \end{aligned}$$

where there are $m-1$ digits (possibly with leading zeroes) and each column can be chosen in three ways (since the only bad configuration would be a zero above a one), which gives 3^{m-1} cases here.

Therefore

$$a_m = 3^{m-1} + a_{m-1},$$

which for us means

$$a_6 = 3^5 + 3^4 + \cdots + 3^1 + a_1 = \frac{3^6-1}{2} + 1 = \boxed{365}.$$ □

Problem 30. **[20]** Compute the number of pairs of natural numbers (a, b) such that $a, b < 1000$ and

$$\frac{a}{b+1} < \sqrt{3} < \frac{a+1}{b}.$$

Note that $\sqrt{3} \approx 1.732$.

Proposed by Daniel Potievsky

Solution 1. For any integer $n > 1$, let $g(n)$ be the greatest positive integer such that $\frac{n}{g(n)} > \sqrt{3}$. In other words,

$$g(n) = \left\lfloor \frac{n}{\sqrt{3}} \right\rfloor.$$

This function g is clearly very closely related to our problem, as it encodes when $\frac{a}{b}$ shifts from below $\sqrt{3}$ to above it. Specifically, we have the following claim:

Claim 30.1 — Call a pair (a, b) of natural numbers *valid* if $\frac{a}{b+1} < \sqrt{3} < \frac{a+1}{b}$, the condition stated in the problem. Assuming that $a > 1$, then (a, b) is valid if and only if $g(a) \le b \le g(a+1)$.

Proof. Suppose $b > g(a+1)$. Then by the way g was defined, $\frac{a+1}{b} < \sqrt{3}$, so (a, b) is not valid. On the other hand, if $b < g(a)$ then $b + 1 \le g(a)$, meaning that $\frac{a}{b+1} > \sqrt{3}$. Again, this means that (a, b) is not valid.

On the other hand, suppose that $g(a) \le b \le g(a+1)$. Then $\frac{a}{b+1}$ must be less than $\sqrt{3}$ and $\frac{a+1}{b}$ must be greater than $\sqrt{3}$, simply by the way g has been defined. □

We know that

$$g(n+1) = \left\lfloor \frac{n+1}{\sqrt{3}} \right\rfloor \ge \left\lfloor \frac{n}{\sqrt{3}} \right\rfloor = g(n)$$

so the interval $[g(a), g(a+1)]$ will always be nonempty; specifically, it will contain $g(a+1) - g(a) + 1$ integers. If $a = 1$ then there is exactly one valid ordered pair, namely $(1, 1)$, so the number of ordered pairs with $1 \le a \le 999$ satisfying the problem statement is

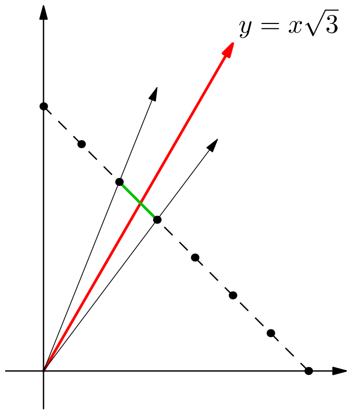
$$1 + \sum_{a=2}^{999} (g(a+1) - g(a) + 1) = 999 + g(1000) - g(2) = 998 + \left\lfloor \frac{1000}{\sqrt{3}} \right\rfloor$$

To compute this quantity, we use the approximation $\sqrt{3} \approx 1.732$ to get

$$\left\lfloor \frac{1000}{\sqrt{3}} \right\rfloor = \left\lfloor \frac{1000\sqrt{3}}{3} \right\rfloor = \left\lfloor \frac{1732}{3} \right\rfloor = 577.$$

The answer is therefore $998 + 577 = \boxed{1575}$. □

Solution 2 (Rishabh Das). Another way of thinking about this problem is to imagine $(b+1, a)$ and $(b, a+1)$ in the coordinate plane: you need $y = \frac{a}{b+1} \cdot x$ to have a slope less than $\sqrt{3}$ and $y = \frac{a+1}{b} \cdot x$ to have a slope greater than $\sqrt{3}$. This means the segment between $(b, a+1)$ and $(b+1, a)$ has to intersect $y = x\sqrt{3}$. If we are given the value of $a + b + 1 = n$, there is only one such choice of a and b, so we need to count the number of possible n.



To find the largest possible value of n, we observe that $1000 > a > b$, so we start by setting $a = 999$. Then b is bounded above by

$$b < \frac{a+1}{\sqrt{3}} = \frac{1000}{\sqrt{3}}.$$

This means the largest possible value of n is

$$999 + \left\lfloor \frac{1000}{\sqrt{3}} \right\rfloor + 1$$

and the smallest is $1 + 1 + 1$. We can also see pretty easily that every n in between gives a valid (a, b), so the total number of n is

$$\left(1000 + \left\lfloor \frac{1000}{\sqrt{3}} \right\rfloor\right) - 3 + 1 = 998 + \left\lfloor \frac{1000}{\sqrt{3}} \right\rfloor.$$

We then compute like in the previous solution to get $\boxed{1575}$. $\square$