New York City Team Contest: Problems

Spring 2023

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1. **[6]** Three identical volumes of calculus with 400 pages each lie upright on a bookshelf next to each other. The thickness of each cover is 5mm, and the thickness of each page is 0.5mm. A worm chews its way, starting between the last page and back cover of the first volume and ending between the front cover and first page of the second volume. Find the distance, in millimeters, traveled by the worm.

   Team Name: ______________ Answer: ______________

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2. **[6]** For positive integers a, b, c, $\sqrt{a}\sqrt[3]{b} = \sqrt{b}\sqrt[3]{c}$. If $\frac{a}{c} = 7$, compute $\frac{a}{b}$.

   Team Name: ______________ Answer: ______________

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3. **[7]** An isosceles trapezoid $ABCD$ with $BC = AD$ has perimeter 42 and smaller base $AB = 3$, the diagonal DB bisects $\angle ABC$. Find the area of $ABCD$.

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4. **[7]** Suppose distinct prime numbers p, q, and r satisfy:

$$pqr - pq - pr + 1 = 2023$$

   Find $p + q + r$.

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5. **[8]** Suppose you have a ten-digit number $N = a_0a_1 \dots a_9$ in base 10 such that for every digit i, a_i is the number of times it appears in N. Find N.

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6. **[8]** Let n be the number of ways there are to split $\{1, 2, \dots, 15, 16\}$ into two disjoint sets A and B such that $|B|$ is minimized and for all elements a in A, either $2a$ or $\frac{a}{2}$ is in B. Find n.

   Team Name: ______________ Answer: ______________

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7. **[9]** Find the largest rational root of a quadratic equation with coefficients that are positive integers less than 2023.

Team Name: ____________ Answer: ____________

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8. **[9]** Let $ABCD$ be a rectangle and X be a point such that X and line segment DC are on opposite sides of AB. Let DX and CX intersect AB at M and N respectively. If the area of DXC is twice the sum of the areas of AMD and BNC, compute the ratio of $XN : NC$.

Team Name: ____________ Answer: ____________

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9. **[10]** 20 people participate in a running league. Every week, for 20 weeks, they race. The person who finishes the race in ith place receives $21 - i$ points. After m weeks, one person has won (meaning that they are guaranteed to finish the season with the most points, no matter how the following weeks unfold). Find the smallest possible value of m.

Team Name: ____________ Answer: ____________

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10. **[Up to 10] Submit a real number between 1 and 10, inclusive.** Your score for this question will be $\frac{2}{5}d(10 - d)$, where d is the absolute difference between your submission and the mean of all submissions for this question.

Team Name: ____________ Answer: ____________

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11. **[11]** How many ways are there to place the numbers $1, 2, \ldots 9$ in a 3×3 grid such that the sums of the numbers in each row and column are all multiples of 3?

Team Name: ____________ Answer: ____________

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12. **[11]** A laser is shot from point A of a rectangle $ABCD$. The laser hits all four sides of the rectangle before hitting side CD again. If CD is the side hit twice, and the laser hits side CD at points P and Q ($CP < CQ$), compute $CP : CQ$.

Team Name: ____________ Answer: ____________

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13. **[12]** Let $x, y, z \geq 0$. Find the minimum value of $x + y + z$ if $(\cos^2 x + \sec^2 x)(1 + \tan^2(2y))(3 + \sin(3z)) = 4$, where all angle measures are in radians.

Team Name: ____________ Answer: ____________

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14. **[12]** Suppose we have rectangle $STAN$ with $ST = 4$ and $TA = 7$. Let M be the midpoint of ST and let E and F be points on TA and SN respectively such that $TENF$ is a parallelogram. Diagonal SA intersects the parallelogram at points P and Q. Given $[MPQ] = 1$, find FN.

Team Name: ____________ Answer: ____________

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15. **[13]** How many positive integers $n < 1000$ have the property that the average of the largest palindrome less than n and the smallest palindrome greater than n is equal to n?

Team Name: ____________ Answer: ____________

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16. **[13]** How many ordered pairs of positive integers (j, p) satisfy:

$$100 \cdot \operatorname{lcm}(j, p) = jp \cdot (j + p)$$

Team Name: ____________ Answer: ____________

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17. **[14]** 7 people of 7 different heights are in a line from front to back. How many ways are there to see exactly 3 of the people in the line, given that you can see someone in the line if and only if they are taller than everyone in front of them?

Team Name: ____________ Answer: ____________

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18. **[14]** Let ΔABC be a triangle with $AB = 20$ and $AC = 23$. Let E be the intersection of the B-angle bisector with AC, and let F be the intersection of the C-angle bisector with AB. Let D be the reflection of A over line EF. If D lies on BC, compute the area of ΔABC.

Team Name: ____________ Answer: ____________

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19. **[15]** Suppose $f(x) = x^2 + 13x + 40$ and let $g(y) = y^2 + 13y + 40$ be its inverse. If P and Q are any points on $f(x)$ and $g(y)$, respectively, find the minimum distance PQ.

Team Name: ____________ Answer: ____________

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20. **[Up to 28]** Welcome to **USAYNO**!

Instructions: Submit a string of 6 letters corresponding to each statement: put T if you think the statement is true, F if you think it is false, and X if you do not wish to answer. You will receive $\frac{(n+1)(n+2)}{2}$ *points for n correct answers, but you will receive zero points if any of the questions you choose to answer are incorrect. Note that this means if you submit "XXXXXX" you will get one point.*

(1) Joseph and Josiah are playing a game. To start, there are the numbers $1, 2, \ldots 12$ written on a chalkboard. A turn consists of circling a previously uncircled number. Joseph and Josiah alternate turns. After each person has completed 3 turns, the median of the circled numbers is calculated. If this number is an integer, Josiah wins. Otherwise, Joseph wins. Given Joseph goes first, there exists a strategy for Joseph to win.

(2) There exists function f from $\mathbb{R} \to \mathbb{R}$ such that for all x, $f(x)^2 + x < 2023f(x)$.

(3) The equation $4a^2b^4 + 2a + b^2 = c^2$ has no solutions in the natural numbers.

(4) Any collection of axial rectangles can be colored using 3 colors such that no two rectangles that share an edge of positive length have the same color.

(5) Consider any circles $\omega_1, \omega_2, \omega_3$ such that ω_2 is tangent to ω_3 at X, ω_3 to ω_1 at Y, and ω_1 to ω_2 at Z. There exists a configuration of ω_1,ω_2,ω_3, and points A,B,C on ω_1, ω_2, and ω_3 such that $\triangle XYZ$ is inscribed in $\triangle ABC$.

(6) It is possible to color the integer lattice with two colors such that there are no squares with 4 vertices of a single color.

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21. **[16]** Find

$$2^{4046} \cdot \sum_{k=1}^{2023} \cos^{4046}\left(\frac{\pi k}{2023}\right)$$

Team Name: _____________ Answer: _____________

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22. **[16]** Let K be a unit tetrahedron. Suppose there exists a sphere ω_1 of radius r and two other spheres ω_2 and ω_3 of radius $\frac{r}{2}$ all enclosed in K and sitting on the base of K. The spheres are all internally tangent to two sides of the tetrahedron and externally tangent to each other. Then, the radius of ω_2 can be written in the form:

$$\frac{\sqrt{a}}{\sqrt{b} + \sqrt{c} + \sqrt{d}}$$

where a, b, c, d are all positive integers for which the sum $a + b + c + d$ is minimized. Find this sum.

Team Name: _____________ Answer: _____________

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23. **[17]** The angle bisector of angle A hits side BC of triangle ABC at point X. The circle tangent to segments CX, and the circumcircle of ABC, and which passes through the incenter of ABC intersects the circumcircle of ABC at the midpoint of arc AC. Given that $BX = 4$ and $CX = 5$, compute the perimeter of triangle ABC.

Team Name: _____________ Answer: _____________

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24. **[17]** Let $p(x)$ be a monic, non-constant polynomial. $p(x)$ satisfies that $p(x^3) = p(x)p(x+1+\omega)p(x+1-\omega)$ for all real x, where $\omega = -\frac{1}{2} + \frac{i\sqrt{3}}{2}$. If $p(3) < 100$, compute the sum of all possible values of $p(4)$.

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25. **[18]** Two paths, only traveling up and to the right, are chosen randomly and independently on a 6×6 grid of points. Both paths originate at the bottom left corner and finish at the top right corner. Find the probability that the paths do not intersect except for at their endpoints.

Team Name: ______________ Answer: ______________

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26. **[18]** Suppose S is the set of all increasing functions $f : \mathbb{R} \longrightarrow \mathbb{R} \setminus \{0\}$ satisfying

$$f(f(n) - \frac{1}{n+1} + 1) = \frac{1}{f(n)}$$

Find

$$\sum_{t=2}^{10} \frac{\sum_S f(t)}{\prod_S f(t)}$$

Team Name: ______________ Answer: ______________