Team Round

1. 75
2. 7
3. -70
4. $\frac{4023}{64}$
5. 3
6. $4+2\sqrt[4]{8}$
7. $\dfrac{2}{7}$
8. $\dfrac{65}{8}$
9. $(4,3),(-3,-4),\left(\dfrac{-13-\sqrt{161}}{2},\dfrac{13-\sqrt{161}}{2}\right),\left(\dfrac{-13+\sqrt{161}}{2},\dfrac{13+\sqrt{161}}{2}\right)$
10. 1015
11. $\dfrac{2\pi\sqrt{3}}{3}$
12. 941192
13. -1
14. $5^{2017}-95$

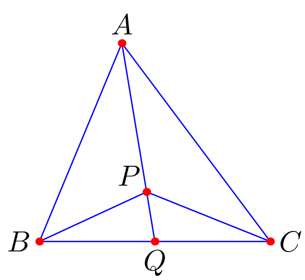
Team Round

1. A team of 6 distinguishable students competed at a math competition. They each scored an integer amount of points, and the sum of their scores was 170. Their highest score was a 29, and their lowest score was a 27. How many possible ordered 6-tuples of scores could they have scored?

   Proposed by Jeffery Li

   Solution: First, note that the only possible ways that the students could've scored if they were indistinguishable were 29,29,29,28,28,27 or 29,29,29,29,27,27. In particular, note that 29,29,28,28,28,28 is not valid since the lowest score is not a 27. For the first case, there are $\frac{6!}{3!2!1!} = 60$ ways, and for the second case, there are $\frac{6!}{4!2!} = 15$ ways. Thus, there are a total of $\boxed{75}$ ways for the students to have scored in such a way.

2. Let ABC be a triangle with $AB = 13$, $BC = 14$, and $CA = 15$. Let P be a point inside triangle ABC, and let ray AP meet segment BC at Q. Suppose the area of triangle ABP is three times the area of triangle CPQ, and the area of triangle ACP is three times the area of triangle BPQ. Compute the length of BQ.



   Proposed by Brandon Wang

   Solution: Let the angle between AQ and BC be $\theta < 90°$. Then, note that $[ABP] = \frac{AP \times BQ \sin\theta}{2}$, $[ACP] = \frac{AP \times CQ \sin\theta}{2}$, $[BPQ] = \frac{PQ \times BQ \sin\theta}{2}$, and $[CPQ] = \frac{PQ \times CQ \sin\theta}{2}$. Now, we have

$$[ABP] = 3[CPQ], [ACP] = 3[BPQ]$$

$$\implies AP \times BQ = 3PQ \times CQ, AP \times CQ = 3PQ \times BQ$$

$$\implies \frac{BQ}{CQ} = \frac{CQ}{BQ} = \frac{3PQ}{AP}$$

$$\implies BQ^2 = CQ^2 \implies BQ = CQ$$

$$\implies BQ = \frac{BC}{2} = \boxed{7}.$$

3. I am thinking of a geometric sequence with 9600 terms, a_1, a_2,…, a_{9600}. The sum of the terms with indices divisible by three (i.e. $a_3 + a_6 + \cdots + a_{9600}$) is $\frac{1}{56}$ times the sum of the other terms (i.e. $a_1 + a_2 + a_4 + a_5 + \cdots + a_{9598} + a_{9599}$). Given that the terms with even indices sum to 10, what is the smallest possible sum of the whole sequence?

   Proposed by Vincent Bian

Solution: If the common ratio is $\frac{1}{r}$, then the ratio of the terms whose indices is divisible by 3 to the rest of the terms is $\frac{1}{r+r^2}$, so $r + r^2 = 56$, meaning r is either 7 or -8.

Note that if the even indices add to 10, then the odd indices must add to $10r$, so the smallest possible sum is $10 - 8 * 10 = \boxed{-70}$.

4. Let $ABCD$ be a regular tetrahedron with side length $6\sqrt{2}$. There is a sphere centered at each of the four vertices, with the radii of the four spheres forming a geometric series with common ratio 2 when arranged in increasing order. If the volume inside the tetrahedron but outside the second largest sphere is 71, what is the volume inside the tetrahedron but outside all four of the spheres?

   Proposed by Sruthi Parthasarathi

   Solution: Note that the volume of $ABCD$ is $\frac{s^3\sqrt{2}}{12} = 72$. Thus, the volume of the region inside both the tetrahedron and the second largest sphere is $72 - 71 = 1$. Since the radii of the sphere form a geometric series with common ratio $\frac{1}{2}$, their volumes form a geometric series with common ratio $\frac{1}{8}$, so the volume common to the tetrahedron and the four spheres are $8, 1, \frac{1}{8}$, and $\frac{1}{64}$. Also, note that none of the spheres have radii greater than $3\sqrt{2}$, or else the overlapping region between that sphere and the tetrahedron contains a tetrahedron of side length $3\sqrt{2}$ and thus that volume is greater than 9, contradiction. Thus, since none of the spheres have radius greater than $3\sqrt{2}$, they don't overlap with each other, so the desired volume is $72 - 8 - 1 - \frac{1}{8} - \frac{1}{64} = \boxed{\frac{4023}{64}}$.

5. Find the largest number of consecutive positive integers, each of which has exactly 4 positive divisors.

   Proposed by Carl Schildkraut

   Solution: Since 33, 34, and 35 each have exactly 4 positive integer divisors, the answer is at least 3. Now, assume for the sake of contradiction that the answer is at least 4. Then, there exists one positive integer divisible by 4 in our list. However, any number $4n$ has the divisors 1,2,4,$2n$, and $4n$, all of which are distinct unless $n = 1$ or 2 (corresponding to 4 and 8). However, 4 only has 3 positive integer divisors, and 8 is not part of a list of 4 consecutive integers that each have 4 positive integer divisors (as 7 has only 2 and 9 has only 3), so the maximum number of these is $\boxed{3}$.

6. Let the (not necessarily distinct) roots of the equation $x^{12} - 3x^4 + 2 = 0$ be $a_1, a_2, \ldots, a_{12}$. Compute

$$\sum_{i=1}^{12} |\text{Re}(a_i)|.$$

   Proposed by Jeffery Li

   Solution: Note that the LHS factors as $(x-1)^2(x+1)^2(x^2+1)^2(x^4+2)$. Thus, the roots are $1, 1, -1, -1, i, i, -i, -i$, and $\frac{\pm\sqrt[4]{2} \pm i\sqrt[4]{2}}{\sqrt{2}} = \pm\frac{\sqrt[4]{8}}{2} \pm i\frac{\sqrt[4]{8}}{2}$. Thus, the sum is equal to $1+1+1+1+4\times\frac{\sqrt[4]{8}}{2} = \boxed{4 + 2\sqrt[4]{8}}$.

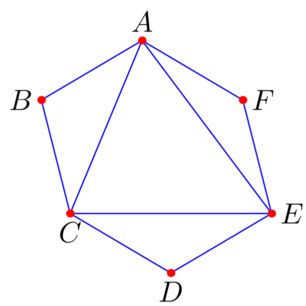
7. Jeffrey is doing a three-step card trick with a row of seven cards labeled A through G. Before he starts his trick, he picks a random permutation of the cards. During each step of his trick, he rearranges the cards in the order of that permutation. For example, for the permutation $(1, 3, 5, 2, 4, 7, 6)$, the first card from the left remains in position, the second card is moved to the third position, the third card

is moved to the fifth position, etc. After Jeffrey completes all three steps, what is the probability that the “A” card will be in the same position as where it started?

Proposed by Eric K. Zhang

Solution: Treat the permutation as a 1-regular functional graph. Then the problem amounts to finding the probability that ”A” is in a cycle of length 1 or 3. The probability it is in a cycle of length one (a loop) is just $\frac{1}{7}$. In the case of a cycle of length three, simply multiply probabilities to get $\left(\frac{6}{7}\right)\left(\frac{5}{6}\right)\left(\frac{1}{5}\right) = \frac{1}{7}$, and add to get $\boxed{\frac{2}{7}}$.

8. In convex equilateral hexagon $ABCDEF$, $AC = 13$, $CE = 14$, and $EA = 15$. It is given that the area of $ABCDEF$ is twice the area of triangle ACE. Compute AB.



Proposed by Jeffery Li

Solution: Let R be the circumradius of ACE. Reflect B over AC to B', D over CE to D', and F over AE to F'. Note that we want $[AB'C] + [CD'E] + [EF'A] = [ACE]$. However, as AB increases, so does $[AB'C] + [CD'E] + [EF'A]$, and when $AB = R$, we will have $B' = D' = F' = O$, where O is the circumcenter of ACE, giving equality. Thus, we must have $AB = R$, and through standard calculations (such as $R = \frac{abc}{4[ABC]}$), we get $AB = \boxed{\frac{65}{8}}$.

9. Find all ordered pairs (x, y) of numbers satisfying

$$(1 + x^2)(1 + y^2) = 170$$
$$(1 + x)(-1 + y) = 10.$$

Proposed by Eric Gan

Solution: Note that the system of equations can be rewritten as:

$$(y - x)^2 + (1 + xy)^2 = 170$$
$$(y - x) + (1 + xy) = 12.$$

Thus, we have

$$\begin{aligned} &(y - x)^2 + 2(y - x)(1 + xy) + (1 + xy)^2 = 144 \\ \implies &(y - x)^2 - 2(y - x)(1 + xy) + (1 + xy)^2 = 196 \\ \implies &(y - x) - (1 + xy) = \pm 14 \\ \implies &y - x = 13, xy = -2 \text{ or } y - x = -1, xy = 12. \end{aligned}$$

Solving the first case gives $(x,y) = \boxed{\left(\frac{-13-\sqrt{161}}{2}, \frac{13-\sqrt{161}}{2}\right), \left(\frac{-13+\sqrt{161}}{2}, \frac{13+\sqrt{161}}{2}\right)}$; solving the second case gives $(x,y) = \boxed{(4,3),(-3,-4)}$. It's easy to check all of these work.

10. Call a positive integer "pretty good" if it is divisible by the product of its digits. Call a positive integer n "clever" if n, $n+1$, and $n+2$ are all pretty good. Find the number of clever positive integers less than 10^{2018}. Note: the only number divisible by 0 is 0.

    Proposed by Sam Ferguson

    Solution: There are exactly 7 single-digit clever integers. It's easy to see that no integer containing a digit 0 is pretty good. Furthermore, any clever integer greater than 9 must have all digits preceding the units digit be 1(s), as for any digit $d \neq 1$, at least one of $n, n+1, n+2$ is not divisible by d. Thus, all clever integers greater than 9 are of the form

    $$\underbrace{111\cdots111}_{m\text{ 1s}}d$$

    for some integer $1 \le m \le 2017$ and some digit d. Since neither $\underbrace{111\cdots111}_{m\text{ 1s}}4$ nor $\underbrace{111\cdots111}_{m\text{ 1s}}8$ are pretty good for any integer $m \ge 1$, clever integers greater than 9 must be of form A: $n = \underbrace{111\cdots111}_{m\text{ 1s}}1$ or form B: $n = \underbrace{111\cdots111}_{m\text{ 1s}}5$. It's easy to see that an integer of form A is clever iff $m = 3k$ for some positive integer k, so there are $\left\lfloor \frac{2017}{3} \right\rfloor = 672$ clever integers of form A less than 10^{2018}.

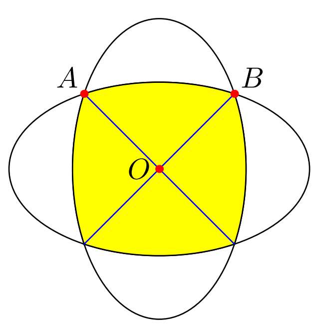
    For form B, clearly n itself is divisible by 5, and $n+1$ is divisible by 6 iff $m = 3k$ for some positive integer k. To test whether $n+2$ is divisible by 7, we rewrite it as $n+2 = \frac{10^{m+1}-1}{9} + 6$. Then $7 \mid n+2$ iff

    $$\frac{10^{m+1}-1}{9} \equiv 1 \mod 7 \iff 10^{m+1} \equiv 10 \mod 7 \iff m \equiv 0 \mod 6$$

    since $10^6 \equiv 1 \mod 7$ but $10^x \not\equiv 1 \mod 7$ for $1 \le x \le 5$. Thus an integer of form B is clever iff $m = 6k$ for some positive integer k, so there are $\left\lfloor \frac{2017}{6} \right\rfloor = 336$ clever integers of form B less than 10^{2018}.

    Thus, there are a total of $7 + 672 + 336 = \boxed{1015}$ clever integers less than 10^{2018}.

11. What is the area in the xy-plane bounded by $x^2 + \frac{y^2}{3} \le 1$ and $\frac{x^2}{3} + y^2 \le 1$?



    Proposed by Vincent Bian

Solution: Note that the ellipses intersect at $\left(\pm\frac{\sqrt{3}}{2}, \pm\frac{\sqrt{3}}{2}\right)$, so let point A be $\left(-\frac{\sqrt{3}}{2}, \frac{\sqrt{3}}{2}\right)$, let B be $\left(\frac{\sqrt{3}}{2}, \frac{\sqrt{3}}{2}\right)$, and let O be the origin. Then, the desired area is made of 4 copies of the elliptical sector AOB (where AOB is a sector of the "horizontal" ellipse defined by $\frac{x^2}{3} + y^2 = 1$)

Now, consider the affine transformation $(x, y) \mapsto (\frac{x}{\sqrt{3}}, y)$, which sends the ellipse $\frac{x^2}{3} + y^2 = 1$ to the unit circle, so AOB gets mapped to a circular sector $A'OB'$. Since A' is at $\left(-\frac{1}{2}, \frac{\sqrt{3}}{2}\right)$ and B' is at $\left(\frac{1}{2}, \frac{\sqrt{3}}{2}\right)$, we have that $\triangle A'OB'$ is equilateral. Thus, sector $A'OB'$ has one sixth of the area of a unit circle, so it has area $\frac{\pi}{6}$.

Since sector $A'OB'$ is $\frac{1}{\sqrt{3}}$ the area of sector AOB, sector AOB has area $\frac{\pi\sqrt{3}}{6}$, and the entire intersection of the ellipses has area $\boxed{\frac{2\pi\sqrt{3}}{3}}$.

12. Let S be the set of ordered triples $(a, b, c) \in \{-1, 0, 1\}^3 \setminus \{(0, 0, 0)\}$. Let n be the smallest positive integer such that there exists a polynomial, with integer coefficients, of the form

$$\sum_{\substack{i+j+k=n \\ i,j,k\geq 0}} a_{(i,j,k)} x^i y^j z^k$$

such that the absolute value of all the coefficients are less than 2, and the polynomial equals 1 for all $(x, y, z) \in S$. Compute the number of such polynomials for that value of n.

Proposed by Jeffery Li

Solution: We will first prove that $n = 6$. First, write the polynomial in terms of x, so that we get something of the form

$$\sum_{i=0}^{n} P_i(y, z) x^i$$

where $P_i(y, z)$ are polynomials in terms of y, z. Plugging in -1 and 1 for x gives us that

$$\sum_{1\leq 2j+1\leq n} P_{2j+1}(y, z) = 0$$

for all relevant ordered pairs (y, z), so we can WLOG assume that no terms of the polynomial have an odd power of x for now. We do something similar for y and z, thus immediately getting that n is even. Now, plugging in $(1, 0, 0)$ and permutations give that the coefficients of x^n, y^n, z^n are all 1. Plugging in $(1, 1, 0)$ and permutations give that

$$\sum_{j=1}^{\frac{n}{2}-1} a_{(2j,n-2j,0)} x^{2j} y^{n-2j} = \sum_{j=1}^{\frac{n}{2}-1} a_{(0,2j,n-2j)} y^{2j} z^{n-2j} = \sum_{j=1}^{\frac{n}{2}-1} a_{(2j,0,n-2j)} x^{2j} z^{n-2j} = -1,$$

and plugging in $(1,1,1)$ gives that

$$\sum_{\substack{i+j+k=\frac{n}{2}\\ i,j,k>0}} a_{(2i,2j,2k)}x^{2i}y^{2j}z^{2k} = 1.$$

Thus, we must have at least one term of the form $x^{2i}y^{2j}z^{2k}$ $(i,j,k>0)$ that has a nonzero coefficient, so $n \geq 2(1+1+1) = 6$. At least one exists with $n=6$; take $x^6+y^6+z^6-x^4y^2-y^4z^2-z^4x^2+x^2y^2z^2$. Thus, we have that $n=6$.

Now, we find the number of such polynomials. From earlier, we must have the coefficients of x^6, y^6, z^6 be 1. Now, consider the terms that only contain x and y, which are x^iy^{6-i} for $1 \leq i \leq 5$. From plugging in $(1,1,0)$ and $(1,-1,0)$, we get that

$$\sum_{i=1}^{5} a_{(i,6-i,0)} = \sum_{i=1}^{5}(-1)^i a_{(i,6-i,0)} = 1$$

$$\implies a_{(1,5,0)} + a_{(3,3,0)} + a_{(5,1,0)} = 0, \quad a_{(2,4,0)} + a_{(4,2,0)} = 1.$$

Since all the terms have absolute value less than 2, there are 7 possible ways to choose the first 3 terms $((0,0,0),(1,-1,0)$ and permutations) and 2 possible ways to choose the last 2 terms $((1,0)$ and $(0,1))$. Thus, there are 14 ways to choose the terms $a_{(i,6-i,0)}$. Similarly, there are 14 ways to choose the terms $a_{(0,i,6-i)}$, and 14 ways to choose the terms $a_{(i,0,6-i)}$.

Now, consider the terms that contain x, y, and z. Denote $A_{xy} = a_{(1,1,4)} + a_{(1,3,2)} + a_{(3,1,2)}$, $A_{xz} = a_{(1,4,1)} + a_{(1,2,3)} + a_{(3,2,1)}$, and $A_{yz} = a_{(4,1,1)} + a_{(2,1,3)} + a_{(2,3,1)}$. Plugging in $(1,1,1),(-1,1,1),(1,-1,1)$, and $(1,1,-1)$, we get

$$\begin{gathered} a_{(2,2,2)} + A_{xy} + A_{xz} + A_{yz} = a_{(2,2,2)} - A_{xy} - A_{xz} + A_{yz} \\ = a_{(2,2,2)} - A_{xy} + A_{xz} - A_{yz} = a_{(2,2,2)} + A_{xy} - A_{xz} - A_{yz} = 1 \\ \implies a_{(2,2,2)} = 1, A_{xy} = A_{yz} = A_{xz} = 0. \end{gathered}$$

This gives us another 7^3 ways to choose the coefficients (We can have $(a_{(1,1,4)}, a_{(1,3,2)}, a_{(3,1,2)})$ equal $(0,0,0)$, $(1,0,-1)$ and permutations, and similar for the terms in A_{xz} and A_{yz}).

Thus, the total number of such polynomials is $14 \times 14 \times 14 \times 7^3 = 98^3 = \boxed{941192}$.

13. Let $N \geq 2017$ be an odd positive integer. Two players, A and B, play a game on an $N \times N$ board, taking turns placing numbers from the set $\{1, 2, \ldots, N^2\}$ into cells, so that each number appears in exactly one cell, and each cell contains exactly one number. Let the largest row sum be M, and the smallest row sum be m. A goes first, and seeks to maximize $\frac{M}{m}$, while B goes second and wishes to minimize $\frac{M}{m}$. There exists real numbers a and $0 < x < y$ such that for all odd $N \geq 2017$, if A and B play optimally,

$$x \cdot N^a \leq \frac{M}{m} - 1 \leq y \cdot N^a.$$

Find a.

Proposed by Brandon Wang

Solution: Let $N = 2k+1$. First, let A place down $\dfrac{N^2+1}{2}$, and then let him do "strategy stealing;" i.e. if there's enough space, then if B places x then let A place N^2+1-x in the same row. This will guarantee that the row sums are between $k(N^2+1)+1$ and $k(N^2+1)+N^2$; thus, since $k = \dfrac{N-1}{2}$, this gives an upper bound of

$$\frac{(N-1)(N^2+1)+2N^2}{(N-1)(N^2+1)+2} = 1 + \frac{2N^2-2}{N^3-N^2+N+1} < 1 + \frac{2}{N-1} < 1 + \frac{c_1}{N}$$

for some constant $c_1 > 0$ and all $N \geq 2017$. B can force one of the row sums to be greater than $k(N^2+1)+N^2-k$ by placing $\frac{N+1}{2}$ of the largest numbers in the same row and making A place $\frac{N-1}{2}$ of the smallest numbers in the same row; since $m \leq \frac{1+2+\cdots+N^2}{N} = \frac{N^3+N}{2}$ we can get a lower bound of

$$\frac{k(N^2+1)+N^2-k}{\frac{N^3+N}{2}} = \frac{N^3+N^2}{N^3+N} = 1 + \frac{N-1}{N^2+1} > 1 + \frac{c_2}{N}$$

for some $c_2 > 0$, $c_2 < c_1$, and all $N \geq 2017$ Thus, $a = \boxed{-1}$.

14. Yunseo has a supercomputer, equipped with a function F that takes in a polynomial $P(x)$ with integer coefficients, computes the polynomial $Q(x) = (P(x)-1)(P(x)-2)(P(x)-3)(P(x)-4)(P(x)-5)$, and outputs $Q(x)$. Thus, for example, if $P(x) = x+3$, then $F(P(x)) = (x+2)(x+1)(x)(x-1)(x-2) = x^5 - 5x^3 + 4x$. Yunseo, being clumsy, plugs in $P(x) = x$ and uses the function 2017 times, each time using the output as the new input, thus, in effect, calculating

$$\underbrace{F(F(F(\ldots F(F(}_{2017}x))\ldots))).$$

She gets a polynomial of degree 5^{2017}. Compute the number of coefficients in the polynomial that are divisible by 5.

Proposed by Jeffery Li

Solution: We work in $\mathbb{F}_5[x]$, or the ring/set of polynomials with coefficients elements of $\mathbb{F}_5$, or integers mod 5. Note that, letting P be shorthand for $P(x)$,

$$F(P) = (P-1)(P-2)(P-3)(P-4)(P-5) = P(P-1)(P-2)(P-3)(P-4) = P^5 - P$$

upon expansion.

Now, define the mapping $\Psi : \mathbb{F}_5[x] \to \mathbb{F}_5[x]$ such that

$$\Psi\left(\sum_{i=1}^{n} a_i x^i\right) = \sum_{i=1}^{n} a_i x^{5^i}.$$

It's easy to see that $\Psi(1) = x$ and Ψ is additive; that is, $\Psi(P+Q) = \Psi(P) + \Psi(Q)$ where $P, Q \in \mathbb{F}_5[x]$. We also have that, if $P(x) = \sum_{i=1}^{n} a_i x^i$, then

$$\Psi(P)^5 = \left(\sum_{i=1}^{n} a_i x^{5^i}\right)^5 = \sum_{i=1}^{n} a_i x^{5^{i+1}} = \Psi(xP)$$

by the Frobenius Endomorphism. Now, we introduce the following:

Lemma: $F^n(x) = \Psi\left((x-1)^n\right)$.

Proof: We use proof by induction. The base case ($n = 0$) is trivial, as $x = \Psi(1)$ is true. Now, suppose that $F^n(x) = \Psi\left((x-1)^n\right)$ for all $n \leq k$. Then,

$$\begin{aligned} F^{k+1}(x) &= F(\Psi\left((x-1)^k\right)) \\ &= \Psi\left((x-1)^k\right)^5 - \Psi\left((x-1)^k\right) \\ &= \Psi\left(x(x-1)^k\right) - \Psi\left((x-1)^k\right) \\ &= \Psi\left((x-1)^{k+1}\right), \end{aligned}$$

completing the inductive step.

Thus, we have that $F^{2017}(x) = \Psi\left((x-1)^{2017}\right)$, so the only coefficients that are nonzero (and thus not divisible by 5) are the ones corresponding to terms of the form x^{5^b} where $\binom{2017}{b} \not\equiv 0 \pmod{5}$. We can count the number of such b by using Lucas' Theorem, which gives us that, if $b = \overline{b_4b_3b_2b_1b_0}_5$ and since $2017 = \overline{31032}_5$, then

$$\binom{2017}{b} \equiv \binom{3}{b_4}\binom{1}{b_3}\binom{0}{b_2}\binom{3}{b_1}\binom{2}{b_0} \pmod{5};$$

thus, for $\binom{2017}{b} \not\equiv 0 \pmod{5}$, we need $0 \le b_4 \le 3, 0 \le b_3 \le 1, b_2 = 0, 0 \le b_1 \le 3, 0 \le b_0 \le 2$, which gives us $4 \times 2 \times 1 \times 4 \times 3 = 96$ such b. Thus, since the rest of the coefficients are zero, we see that $(5^{2017}+1) - 96 = \boxed{5^{2017} - 95}$ coefficients are divisible by 5.

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