HMMT November 2019

November 9, 2019

Team Round

1. **[20]** Each person in Cambridge drinks a (possibly different) 12 ounce mixture of water and apple juice, where each drink has a positive amount of both liquids. Marc McGovern, the mayor of Cambridge, drinks $\frac{1}{6}$ of the total amount of water drunk and $\frac{1}{8}$ of the total amount of apple juice drunk. How many people are in Cambridge?

   Proposed by: Alec Sun

   Answer: $\boxed{7}$

   The total amount of liquid drunk must be more than 6 times of the amount that Marc drinks but less than 8 times of the amount that Marc drinks. Therefore, the number of people in Cambridge is more than 6 but less than 8, so it must be 7.

2. **[20]** 2019 students are voting on the distribution of N items. For each item, each student submits a vote on who should receive that item, and the person with the most votes receives the item (in case of a tie, no one gets the item). Suppose that no student votes for the same person twice. Compute the maximum possible number of items one student can receive, over all possible values of N and all possible ways of voting.

   Proposed by: Milan Haiman

   Answer: $\boxed{1009}$

   To get an item, a student must receive at least 2 votes on that item. Since each student receives at most 2019 votes, the number of items one student can receive does not exceed $\frac{2019}{2} = 1009.5$. So, the answer is at most 1009. This occurs when $N = 2018$ and item i was voted to student $1, 1, 2, 3, \ldots, 2018$ by student $2i-1, 2i-2, \ldots, 2019, 1, \ldots, 2i-2$ respectively for $i = 1, 2, \ldots, 2018$. Thus, the maximum possible number of items one student can receive is 1009.

3. **[30]** The coefficients of the polynomial $P(x)$ are nonnegative integers, each less than 100. Given that $P(10) = 331633$ and $P(-10) = 273373$, compute $P(1)$.

   Proposed by: Carl Joshua Quines

   Answer: $\boxed{100}$

   Let
   $$P(x) = a_0 + a_1 x + a_2 x^2 + \ldots$$
   Then
   $$\frac{1}{2}(P(10) + P(-10)) = a_0 + 100a_2 + \ldots$$
   and
   $$\frac{1}{2}(P(10) - P(-10)) = 10a_1 + 1000a_3 + \ldots$$
   Since all the coefficients are nonnegative integers, these expressions give us each of the coefficients by just taking two digits in succession. Thus we have $a_0 = 3$, $a_1 = 13$, $a_2 = 25$, $a_3 = 29$, $a_4 = 30$ and $a_n = 0$ for $n > 4$. Thus
   $$P(1) = a_0 + a_1 + a_2 + \cdots = 100.$$

4. **[35]** Two players play a game, starting with a pile of N tokens. On each player's turn, they must remove 2^n tokens from the pile for some nonnegative integer n. If a player cannot make a move, they lose. For how many N between 1 and 2019 (inclusive) does the first player have a winning strategy?

   Proposed by: Milan Haiman

   Answer: $\boxed{1346}$

   The first player has a winning strategy if and only if N is not a multiple of 3. We show this by induction on N. If $N = 0$, then the first player loses.

If N is a multiple of 3, then $N - 2^n$ is never a multiple of 3 for any n, so the second player has a winning strategy.

If N is not a multiple of 3, the first player can remove either 1 or 2 coins to get the number of coins in the pile down to a multiple of 3, so the first player will always win.

5. **[40]** Compute the sum of all positive real numbers $x \leq 5$ satisfying

$$x = \frac{\lceil x^2 \rceil + \lceil x \rceil \cdot \lfloor x \rfloor}{\lceil x \rceil + \lfloor x \rfloor}.$$

Proposed by: Milan Haiman

Answer: $\boxed{85}$

Note that all integer x work. If x is not an integer then suppose $n < x < n+1$. Then $x = n + \frac{k}{2n+1}$, where n is an integer and $1 \leq k \leq 2n$ is also an integer, since the denominator of the fraction on the right hand side is $2n+1$. We now show that all x of this form work.

Note that

$$x^2 = n^2 + \frac{2nk}{2n+1} + \left(\frac{k}{2n+1}\right)^2 = n^2 + k - \frac{k}{2n+1} + \left(\frac{k}{2n+1}\right)^2.$$

For $\frac{k}{2n+1}$ between 0 and 1, $-\frac{k}{2n+1} + \left(\frac{k}{2n+1}\right)^2$ is between $-\frac{1}{4}$ and 0, so we have $n^2 + k - 1 < x^2 \leq n^2 + k$, and $\lceil x^2 \rceil = n^2 + k$.

Then,

$$\frac{\lceil x^2 \rceil + \lceil x \rceil \cdot \lfloor x \rfloor}{\lceil x \rceil + \lfloor x \rfloor} = \frac{n^2 + k + n \cdot (n+1)}{2n+1} = n + \frac{k}{2n+1} = x,$$

so all x of this form work.

Now, note that the $2n$ solutions in the interval $(n, n+1)$, together with the solution $n+1$, form an arithmetic progression with $2n+1$ terms and average value $n + \frac{n+1}{2n+1}$. Thus, the sum of the solutions in the interval $(n, n+1]$ is $2n^2 + 2n + 1 = n^2 + (n+1)^2$. Summing this for n from 0 to 4, we get that the answer is

$$0^2 + 2(1^2 + 2^2 + 3^2 + 4^2) + 5^2 = 85.$$

6. **[45]** Let $ABCD$ be an isosceles trapezoid with $AB = 1$, $BC = DA = 5$, $CD = 7$. Let P be the intersection of diagonals AC and BD, and let Q be the foot of the altitude from D to BC. Let PQ intersect AB at R. Compute $\sin \angle RPD$.

Proposed by: Milan Haiman

Answer: $\boxed{\frac{4}{5}}$

Let M be the foot of the altitude from B to CD. Then $2CM + AB = CD \implies CM = 3$. Then $DM = 4$ and by the Pythagorean theorem, $BM = 4$. Thus BMD is a right isosceles triangle i.e. $\angle BDM = \angle PDC = \frac{\pi}{4}$. Similarly, $\angle PCD = \frac{\pi}{4}$. Thus $\angle DPC = \frac{\pi}{2}$, which means quadrilateral $PQDC$ is cyclic. Now, $\sin \angle RPD = \sin \angle DCQ = \sin \angle MCB = \frac{4}{5}$.

An alternate solution is also possible:

Note that $AC \perp BD$ since $AB^2 + CD^2 = 1^2 + 7^2 = 5^2 + 5^2 = BC^2 + DA^2$. Thus P is the foot of the altitude from D to AC. Since D is on the circumcircle of $\triangle ABC$, line PQR is the Simson line of D. Thus R is the foot from D to AB. Then from quadrilateral $RAPD$ being cyclic we have $\angle RPD = \angle RAD$. So $\sin \angle RPD = \frac{4}{5}$.

7. **[55]** Consider sequences a of the form $a = (a_1, a_2, \ldots, a_{20})$ such that each term a_i is either 0 or 1. For each such sequence a, we can produce a sequence $b = (b_1, b_2, \ldots, b_{20})$, where

$$b_i = \begin{cases} a_i + a_{i+1} & i = 1 \\ a_{i-1} + a_i + a_{i+1} & 1 < i < 20 \\ a_{i-1} + a_i & i = 20. \end{cases}$$

How many sequences b are there that can be produced by more than one distinct sequence a?

Proposed by: Benjamin Qi

Answer: $\boxed{64}$

Let the two sequences be b and $\hat{b}$. Then, observe that given a, if $b_1 = \hat{b}_1$ and $b_2 = \hat{b}_2$, then $b = \hat{b}$ (since a will uniquely determine the remaining elements in b and $\hat{b}$). Thus, b and $\hat{b}$ must start with $(1, 0, \dots)$ and $(0, 1, \dots)$, respectively (without loss of generality).

Note that a_3 is either 1 (in which case $b_3 = \hat{b}_3 = 0$) or 2 (in which case $b_3 = \hat{b}_3 = 1$). Moreover, b_4, b_5 must be the same as b_1, b_2 (and the same for $\hat{b}$) for the sequences to generate the same a_3, a_4. We can then pick $a_6, a_9, \dots$.

Observe, that the last elements also have to be $(\dots, 1, 0)$ for b and $(\dots, 0, 1)$ for $\hat{b}$. Thus, the answer is nonzero only for sequence lengths of $2 \pmod 3$, in which case, our answer is 2^k, where the length is $3k+2$ (since we have two choices for every third element).

Here, since $N = 20 = 3(6) + 2$, the answer is $2^6 = 64$.

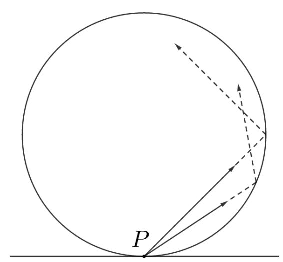
8. **[60]** In $\triangle ABC$, the external angle bisector of $\angle BAC$ intersects line BC at D. E is a point on ray $\overrightarrow{AC}$ such that $\angle BDE = 2\angle ADB$. If $AB = 10$, $AC = 12$, and $CE = 33$, compute $\frac{DB}{DE}$.

Proposed by: Milan Haiman

Answer: $\boxed{\frac{2}{3}}$

Let F be a point on ray $\overrightarrow{CA}$ such that $\angle ADF = \angle ADB$. $\triangle ADF$ and $\triangle ADB$ are congruent, so $AF = 10$ and $DF = DB$. So, $CF = CA + AF = 22$. Since $\angle FDC = 2\angle ADB = \angle EDC$, by the angle bisector theorem we compute $\frac{DF}{DE} = \frac{CF}{CE} = \frac{22}{33} = \frac{2}{3}$.

9. **[65]** Will stands at a point P on the edge of a circular room with perfectly reflective walls. He shines two laser pointers into the room, forming angles of n° and $(n+1)^\circ$ with the tangent at P, where n is a positive integer less than 90. The lasers reflect off of the walls, illuminating the points they hit on the walls, until they reach P again. (P is also illuminated at the end.) What is the minimum possible number of illuminated points on the walls of the room?



Proposed by: Handong Wang

Answer: $\boxed{28}$

Note that we want the path drawn out by the lasers to come back to P in as few steps as possible. Observe that if a laser is fired with an angle of n degrees from the tangent, then the number of points it creates on the circle is $\frac{180}{\gcd(180,n)}$. (Consider the regular polygon created by linking all the points that show up on the circle–if the center of the circle is O, and the vertices are numbered $V_1, V_2, \dots, V_k$, the angle $\angle V_1 O V_2$ is equal to $2\gcd(180, n)$, so there are a total of $\frac{360}{2\gcd(180,n)}$ sides).

Now, we consider the case with both n and $n+1$. Note that we wish to minimize the value $\frac{180}{\gcd(180,n)} + \frac{180}{\gcd(180,n+1)}$, or maximize both $\gcd(180, n)$ and $\gcd(180, n+1)$. Note that since n and $n+1$ are relatively prime and $180 = (4)(9)(5)$, the expression is maximized when $\gcd(180, n) = 20$ and $\gcd(180, n+1) = 9$ (or vice versa). This occurs when $n = 80$. Plugging this into our expression, we have that the number of points that show up from the laser fired at 80 degrees is $\frac{180}{20} = 9$ and the number of points that appear from the laser fired at 81 degrees is $\frac{180}{9} = 20$. However, since both have a point that shows up at P (and no other overlapping points since $\gcd(9, 20) = 1$), we see that the answer is $20 + 9 - 1 = 28$.

10. **[70]** A convex 2019-gon $A_1A_2\dots A_{2019}$ is cut into smaller pieces along its 2019 diagonals of the form A_iA_{i+3} for $1 \le i \le 2019$, where $A_{2020} = A_1$, $A_{2021} = A_2$, and $A_{2022} = A_3$. What is the least possible number of resulting pieces?

Proposed by: Krit Boonsiriseth

Answer: $\boxed{5049}$

Each time we draw in a diagonal, we create one new region, plus one new region for each intersection on that diagonal. So, the number of regions will be

$$1 + (\text{number of diagonals}) + (\text{number of intersections}),$$

where (number of intersections) counts an intersection of three diagonals twice. Since no four diagonals can pass through a point, the only nonconstant term in our expression is the last one. To minimize this term, we want to maximize the number of triples of diagonals passing through the same point. Consider the set S of triples of diagonals A_nA_{n+3} that intersect at a single point. Each triple in S must come from three consecutive diagonals, and two different triples can only have one diagonal in common, so S has at most $\left\lfloor \frac{2019}{2} \right\rfloor = 1009$ triples. Hence the number of resulting pieces is at least

$$1 + (2019) + (2 \cdot 2019 - 1009) = 5049.$$

To show that 5049 is attainable, we use the following construction. Let $B_1 \dots B_{1010}$ be a regular 1010-gon, and let ℓ_n denote the external angle bisector of $\angle B_{n-1}B_nB_{n+1}$. Let $A_1 = \overleftrightarrow{B_{1009}B_{1010}} \cap \overleftrightarrow{B_1B_2}$, $A_{2018} = \overleftrightarrow{B_{1008}B_{1009}} \cap \overleftrightarrow{B_{1010}B_1}$, $A_{2019} = \ell_1 \cap \ell_{1009}$, and for $n = 1, \dots, 1008$, define $A_{2n} = \ell_{n+1} \cap \overleftrightarrow{B_{n-1}B_n}$ and $A_{2n+1} = \ell_n \cap \overleftrightarrow{B_{n+1}B_{n+2}}$. It follows that, for all $n = 0, \dots, 1008$, $\overline{A_{2n-1}A_{2n+2}}$, $\overline{A_{2n}A_{2n+3}}$, and $\overline{A_{2n+1}A_{2n+4}}$ intersect at B_{n+1}.