HMMT November 2019

November 9, 2019

Team Round

1. **[20]** Each person in Cambridge drinks a (possibly different) 12 ounce mixture of water and apple juice, where each drink has a positive amount of both liquids. Marc McGovern, the mayor of Cambridge, drinks $\frac{1}{6}$ of the total amount of water drunk and $\frac{1}{8}$ of the total amount of apple juice drunk. How many people are in Cambridge?

2. **[20]** 2019 students are voting on the distribution of N items. For each item, each student submits a vote on who should receive that item, and the person with the most votes receives the item (in case of a tie, no one gets the item). Suppose that no student votes for the same person twice. Compute the maximum possible number of items one student can receive, over all possible values of N and all possible ways of voting.

3. **[30]** The coefficients of the polynomial $P(x)$ are nonnegative integers, each less than 100. Given that $P(10) = 331633$ and $P(-10) = 273373$, compute $P(1)$.

4. **[35]** Two players play a game, starting with a pile of N tokens. On each player's turn, they must remove 2^n tokens from the pile for some nonnegative integer n. If a player cannot make a move, they lose. For how many N between 1 and 2019 (inclusive) does the first player have a winning strategy?

5. **[40]** Compute the sum of all positive real numbers $x \leq 5$ satisfying

$$x = \frac{\lceil x^2 \rceil + \lceil x \rceil \cdot \lfloor x \rfloor}{\lceil x \rceil + \lfloor x \rfloor}.$$

6. **[45]** Let $ABCD$ be an isosceles trapezoid with $AB = 1$, $BC = DA = 5$, $CD = 7$. Let P be the intersection of diagonals AC and BD, and let Q be the foot of the altitude from D to BC. Let PQ intersect AB at R. Compute $\sin \angle RPD$.

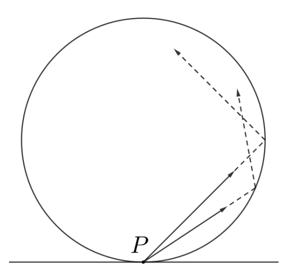
7. **[55]** Consider sequences a of the form $a = (a_1, a_2, \ldots, a_{20})$ such that each term a_i is either 0 or 1. For each such sequence a, we can produce a sequence $b = (b_1, b_2, \ldots, b_{20})$, where

$$b_i = \begin{cases} a_i + a_{i+1} & i = 1 \\ a_{i-1} + a_i + a_{i+1} & 1 < i < 20 \\ a_{i-1} + a_i & i = 20. \end{cases}$$

How many sequences b are there that can be produced by more than one distinct sequence a?

8. **[60]** In $\triangle ABC$, the external angle bisector of $\angle BAC$ intersects line BC at D. E is a point on ray $\overrightarrow{AC}$ such that $\angle BDE = 2\angle ADB$. If $AB = 10$, $AC = 12$, and $CE = 33$, compute $\frac{DB}{DE}$.

9. **[65]** Will stands at a point P on the edge of a circular room with perfectly reflective walls. He shines two laser pointers into the room, forming angles of n° and $(n+1)^\circ$ with the tangent at P, where n is a positive integer less than 90. The lasers reflect off of the walls, illuminating the points they hit on the walls, until they reach P again. (P is also illuminated at the end.) What is the minimum possible number of illuminated points on the walls of the room?



10. **[70]** A convex 2019-gon $A_1A_2 \ldots A_{2019}$ is cut into smaller pieces along its 2019 diagonals of the form A_iA_{i+3} for $1 \leq i \leq 2019$, where $A_{2020} = A_1$, $A_{2021} = A_2$, and $A_{2022} = A_3$. What is the least possible number of resulting pieces?