NYCMT 2023-2024 Homework #6 Solutions

NYCMT

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1 Same Old, Same Old

Problem 1. A fair four-sided die has faces labeled 1, 2, 3, and 4. This die is rolled six times, and the numbers rolled are multiplied together. The probability that this product is a perfect square can be expressed as $\frac{m}{n}$, where m and n are relatively prime positive integers. Find $m + n$.

Answer. $\boxed{161}$

Solution 1 - Casework. We will count the number of sequences of six rolls that produce a perfect square product, and then divide by the total number of possible sequences, 4^6.

Note that rolling a 1 or a 4 does not affect whether the product is a perfect square, so we need to roll an even number of 2s and an even number of 3s. We now do casework on the number of times a perfect square is rolled.

Case 1: A perfect square is never rolled. Then, we need an even number of 2s rolled and an even number of 3s rolled, which could play out as 0 and 6, 2 and 4, 4 and 2, or 6 and 0 rolls, respectively. This results in $\binom{6}{0} + \binom{6}{2} + \binom{6}{4} + \binom{6}{6} = 32$ valid sequences.

Case 2: A perfect square is rolled twice. There are $\binom{6}{2}$ ways to place the squares, and 2^2 choices for the squares. Then, there could be 0 and 4, 2 and 2, or 4 and 0 2s and 3s rolled, which results in $\binom{6}{2} \cdot 2^2 \cdot \left(\binom{4}{0} + \binom{4}{2} + \binom{4}{4}\right) = 15 \cdot 4 \cdot 8 = 480$ valid sequences.

Case 3: A perfect square is rolled four times. There are $\binom{6}{4}$ ways to place the squares, and 2^4 choices for the squares. Then, there could be 0 and 2 or 2 and 0 2s and 3s rolled, which results in $\binom{6}{4} \cdot 2^4 \cdot \left(\binom{2}{0} + \binom{2}{2}\right) = 15 \cdot 16 \cdot 2 = 480$ valid sequences.

Case 4: A perfect square is rolled six times. There are clearly $2^6 = 64$ valid sequences.

The desired probability is then $\frac{32+480+480+64}{4^6} = \frac{1056}{4096} = \frac{33}{128}$, and $m + n = 33 + 128 = \boxed{161}$. $\square$

Solution 2 - Generating Functions. Prime factorizing each of the faces gives 1, 2, 3, and 2^2. We define the generating function $F(p,q) = (1+p+q+p^2)^6$ to represent rolling the four-sided die six times. With this construction, the expansion of F contains terms of the form ap^iq^j, where a represents the number of ways to achieve a product of 2^i3^j.

With this, we can easily see the prime factorization of each possible product. This allows us to determine whether a product is a perfect square; namely, when i and j are both even. So, we want the sum of all coefficients of terms ax^iy^j, where i and j are both even. Then, we will divide this by the total number of possible rollings, 4^6, to get our final probability.

We claim that the desired sum is equal to $\frac{1}{4}\left(F(-1,-1)+F(-1,1)+F(1,-1)+F(1,1)\right)$, or the average of F evaluated at all permutations of -1s and 1s. We can see that, for some term ap^iq^j, if i and j are both even, then plugging in $(p,q) = (\pm 1, \pm 1)$ always produces a, so we get $\frac{1}{4}(4a) = a$, as desired. If at least one of i and j are odd, WLOG let i be odd, then $F(-1,-1) = -F(1,-1)$, and $F(-1,1) = -F(1,1)$, and symmetrically if j is odd. This means that all other terms cancel out to zero, and we get exactly what we want.

The desired probability is then

$$\frac{\frac{1}{4}\left(F(-1,-1)+F(-1,1)+F(1,-1)+F(1,1)\right)}{4^6} = \frac{0^6+2^6+2^6+4^6}{4^7} = \frac{33}{128}.$$

The answer is $m+n = 33+128 = \boxed{161}$.

Remark. This approach can be generalized to any number of rolls of any di(c)e. □

Problem 2. There exists a unique positive integer n such that $36n$ has 36 factors and $48n$ has 48 factors. How many factors does n^2 have?

Answer. $\boxed{39}$

Solution. Let $\tau(n)$ denote the number of positive integer factors of n. We will work with the weaker condition

$$\tau(48n) = \frac{4}{3} \cdot \tau(36n)$$

which is clearly true, as $48 = \frac{4}{3} \cdot 36$.

We can express n as $2^a \cdot 3^b \cdot x$, where a and b are the exponents of 2 and 3 in the prime factorization of n. (This means that x is relatively prime to 2 and 3.)

Now, $\tau(36n) = \tau\left(2^{a+2} \cdot 3^{b+2} \cdot x\right) = (a+3)(b+3) \cdot \tau(x)$, because τ is multiplicative. Similarly, $\tau(48n) = \tau\left(2^{a+4} \cdot 3^{b+1} \cdot x\right) = (a+5)(b+2) \cdot \tau(x)$. Dividing this equation by the first gives

$$\frac{\tau(48n)}{\tau(36n)} = \frac{4}{3} = \frac{a+5}{a+3} \cdot \frac{b+2}{b+3}.$$

Recall that a and b are non-negative integers. We now do casework on the value of a.

If $a = 0$, then $\frac{5}{3} \cdot \frac{b+2}{b+3} = \frac{4}{3}$, so $b = 2$. We also have $\tau(36n) = (0+3)(2+3) \cdot \tau(x) = 15\tau(x) = 36$. This means $\tau(x) = \frac{12}{5}$, which is impossible, as $\tau(x)$ must be a positive integer.

If $a = 1$, then $\frac{6}{4} \cdot \frac{b+2}{b+3} = \frac{4}{3}$, so $b = 6$. We also have $\tau(36n) = (1+3)(6+3) \cdot \tau(x) = 36\tau(x) = 36$. This means $\tau(x) = 1$, so $x = 1$.

If $a = 2$, then $\frac{7}{5} \cdot \frac{b+2}{b+3} = \frac{4}{3}$, so $b = 18$. We also have $\tau(36n) = (2+3)(18+3) \cdot \tau(x) = 105\tau(x) = 36$. This means $\tau(x) = \frac{12}{35}$, which is impossible, as $\tau(x)$ must be a positive integer.

If $a \geq 3$, then $\frac{a+5}{a+3} \leq \frac{4}{3}$, and no non-negative values of b work, because $\frac{b+2}{b+3}$ is always less than 1.

We see that the only valid case is when $a = 1$, $b = 6$, and $x = 1$, which means $n = 2 \cdot 3^6$. Then, $n^2 = 2^2 \cdot 3^{12}$, which has $3 \cdot 13 = \boxed{39}$ factors. □

Problem 3. In $\triangle ABC$, D is the midpoint of $\overline{BC}$, E is the midpoint of $\overline{AC}$, and F is the midpoint of $\overline{AB}$. If $\overline{AD} \perp \overline{BE}$, $AD = 63$, and $BE = 84$, find the length of $\overline{CF}$.

Answer. $\boxed{105}$

Solution. Let G be the centroid of $\triangle ABC$. We know that $\frac{AG}{GD} = \frac{BG}{GE} = 2$, so $AG = 42$, $GD = 21$, $BG = 56$, and $GE = 28$. $\triangle AGB$ is a right triangle, which means F is its circumcenter and $FG = FA = FB = \frac{1}{2}AB$. We can find $AB = \sqrt{AG^2 + BG^2} = \sqrt{42^2 + 56^2} = 70$ by the Pythagorean Theorem, and $FG = \frac{1}{2}AB = 35$. And, because $\frac{CG}{GF} = 2$, we have $CG = 70$ and $CF = \boxed{105}$.

□

Problem 4. Chenkai and Edwin are eating croffles at constant rates. Chenkai can eat an entire croffle by himself in 45 seconds, but Edwin eats even faster, and can do so in x seconds. Chenkai and Edwin each start eating their own croffle at the same time, but when Edwin is done with his, Chenkai realizes that he is eating someone else's croffle. After apologizing profusely, he buys another. Edwin is still hungry, so they both eat the third croffle. If Edwin spent a total of 56 seconds eating croffles, find x.

Answer. $\boxed{36}$

Solution. The number of seconds Edwin spent eating his first croffle is simply x. The third croffle is eaten by both Chenkai and Edwin, who have croffle-eating rates of $\frac{1}{45}$ and $\frac{1}{x}$ of a croffle per second, respectively. The sum of these rates is $\frac{x+45}{45x}$, and the time taken to eat the third croffle is its reciprocal.

We now have the following equation:

$$\begin{aligned} x + \frac{45x}{x+45} &= 56 \\ x^2 + 45x + 45x &= 56x + 2520 \\ x^2 + 34x - 2520 &= 0 \\ (x-36)(x+70) &= 0 \end{aligned}$$

so $x = \boxed{36}$. $\square$

Problem 5. Let $\lfloor x \rfloor$ denote the greatest integer less than or equal to x. Find the ninth smallest positive integer n such that there are no real solutions to $x \cdot \lfloor x \rfloor = n$.

Answer. $\boxed{90}$

Solution. We will first build the set of all real numbers that can be expressed as $x \cdot \lfloor x \rfloor$ for some real number x by doing casework on the value of the integer $\lfloor x \rfloor$.

Case 1: $\lfloor x \rfloor = 0$. Then, $x \cdot \lfloor x \rfloor$ is clearly 0.

Case 2: $\lfloor x \rfloor = k > 0$. Then, $k \leq x < k+1$, which means that $k^2 \leq x \cdot \lfloor x \rfloor < k^2 + k$. This achieves every value in the interval $[k^2, k^2+k)$ for each positive integer k. The first three such intervals are $[1,2)$, $[4,6)$, and $[9,12)$.

Case 3: $\lfloor x \rfloor = k < 0$. Like in Case 2, $k \leq x < k+1$, but now, $\lfloor x \rfloor$ is negative, so multiplying will reverse the inequality: $k^2 \geq x \cdot \lfloor x \rfloor > k^2 + k$. Importantly, this is still a single interval, because $k < 0$. To see this more clearly, we can let $l = -k$ and rewrite the inequality as $l^2 - l < x \cdot \lfloor x \rfloor \leq l^2$. Then, since k ranges over all negative integers, l ranges over all positive integers. This achieves every value in the interval $(l^2 - l, l^2]$ for each positive integer l. The first three such intervals are $(0,1]$, $(2,4]$, and $(6,9]$.

We have exhausted all possible values of $\lfloor x \rfloor$, so we now take the union of all possible values of $x \cdot \lfloor x \rfloor$, which gives

$$\begin{aligned}
&\{0\} \cup \bigcup_{k=1}^{\infty} \left[k^2, k^2+k\right) \cup \bigcup_{l=1}^{\infty} \left(l^2 - l, l^2\right] \\
&= \{0\} \cup \bigcup_{m=1}^{\infty} \left((m^2 - m, m^2] \cup [m^2, m^2+m)\right) \\
&= \{0\} \cup \bigcup_{m=1}^{\infty} \left(m^2 - m, m^2 + m\right) \\
&= \{0\} \cup (0,2) \cup (2,6) \cup (6,12) \cup (12,20) \cup \dots \\
&= [0, \infty) \backslash \{2, 6, 12, \dots\}
\end{aligned}$$

It looks like the set of achievable numbers is the non-negative reals, not including a specific set of positive integers. Indeed, the right endpoint of the mth interval is equal to the left endpoint of the $(m+1)$th interval, because $m^2 + m = (m+1)^2 - (m+1)$. And because each interval is open, the only values that are not achievable are these endpoints, of the form $m^2 + m$ for each positive integer m.

The mth smallest positive integer that is not achievable is $m^2 + m$, so the ninth such integer is $9^2 + 9 = \boxed{90}$. $\square$

2 Circular Reasoning

We will first solve most of the problems in terms of the unknown variable, and then casework on the value of B.

Problem 6. Let F be the answer to Problem 11. Find the remainder when

$$F + 2F + 3F + \cdots + F^2$$

is divided by 100.

Solution. There is not much we can do in this problem without knowing the value of F. There is, however, a crucial restriction. We are looking for the remainder when a positive integer value is divided by 100, so the answer must satisfy $\boxed{0 \le A < 100}$. This fact will be important later. $\square$

Problem 7. Let A be the answer to Problem 6. On Day 0, there are A bananas. On each subsequent day, if there are k bananas left, Noam eats $\lceil \frac{k}{2} \rceil$ of them. On what day does Noam eat the last banana?

Solution. We note that $k - \lceil \frac{k}{2} \rceil = \lfloor \frac{k}{2} \rfloor$. If k is even, then $\lfloor \frac{k}{2} \rfloor = \frac{k}{2}$, and if k is odd, then $\lfloor \frac{k}{2} \rfloor = \frac{k-1}{2}$. Now, consider the binary representation of k. If k is even, then it ends in a 0, and removing that 0 produces $\frac{k}{2} = \lfloor \frac{k}{2} \rfloor$. If k is odd, then it ends in a 1, and removing that 1 produces $\frac{k-1}{2} = \lfloor \frac{k}{2} \rfloor$. This means that, each day, the number of bananas loses the units digit of its binary representation. So, the answer to this problem is the number of digits in the binary representation of A, which can be computed as follows: $\boxed{B = \lfloor \log_2 A \rfloor + 1}$. $\square$

Problem 8. Let B be the answer to Problem 7. There are B friends who want to participate in a game of assassin, in which each player is assigned a target to eliminate. When a player is eliminated, their target becomes the new target of their eliminator. (Players cannot eliminate themselves.) Once there are no more possible eliminations, there is one player left. How many initial assignments of targets were possible?

Remark. This problem was clarified to include the constraint that players' assigned targets are pairwise distinct.

Solution. In order to guarantee one player left at the end of the game, the order of targets must be a single loop containing all players. (If there are two or more disjoint loops, then each loop will have one player that cannot eliminate players from other loops.) Now, consider an arbitrary friend. There are $B - 1$ possibilities for their target, $B - 2$ possibilities for their target's target, and so on. The answer is then $\boxed{C = (B - 1)!}$. $\square$

Problem 9. Let C be the answer to Problem 8. A right triangle with perimeter C has side lengths that form an arithmetic progression. Find the sum of the lengths of its legs.

Solution. Let the lengths of the sides of this right triangle be x, $x+k$, and $x+2k$, where $x, k > 0$. By the Pythagorean Theorem,

$$\begin{aligned} x^2 + (x+k)^2 &= (x+2k)^2 \\ 2x^2 + 2xk + k^2 &= x^2 + 4xk + 4k^2 \\ x^2 - 2xk - 3k^2 &= 0 \\ (x-3k)(x+k) &= 0 \end{aligned}$$

This means that $x = -k$ or $x = 3k$. The former is impossible, so we can use the latter to rewrite the sides as $3k$, $4k$, and $5k$. The perimeter is $12k = C$, and the sum of the legs is $7k = \boxed{D = \frac{7}{12}C}$. □

Problem 10. Let D be the answer to Problem 9. A rectangle with area D has diagonals that intersect at an angle θ, with $\sin\theta = \frac{1}{7}$. This rectangle is inscribed in a circle of radius r. Find r.

Solution. The diagonals divide the rectangle's area into four isosceles triangles, each of which have legs of length r and vertex angle θ or $180° - \theta$. Using the fact that $\sin(\theta) = \sin(180° - \theta)$, the area of each triangle is $K = \frac{1}{2} \cdot r \cdot r \cdot \sin(\theta)$, so the area of the rectangle is $D = 4K = \frac{2}{7}r^2$, and $r = \boxed{E = \sqrt{\frac{7}{2}D}}$. □

Problem 11. Let E be the answer to Problem 10. Rishabh has E colors of socks, and E socks of each color, for a total of E^2 socks in his drawer. He removes socks one at a time until he is absolutely certain that he has at least one sock of each color. How many socks are left in the drawer?

Solution. We aim to maximize F, the number of socks left in the drawer, while still guaranteeing that Rishabh has removed at least one sock of each color. If $F \geq E$, then it is possible that all E socks of a single color are still in the drawer, and the condition is not guaranteed. If, however, $F < E$, it is no longer possible that all E socks of a single color are still in the drawer, meaning Rishabh must have removed at least one of each color. The maximum $F < E$ is then $\boxed{F = E - 1}$. □

Putting it all together. We now aim to solve for the values of A, B, C, D, E, and F. First, we know that B is always an integer, because $\lfloor \log_2 A \rfloor$ is always an integer, no matter what A is. (We know $A > 0$ because Problem 7 implies that Noam eats at least one banana, so B is always defined.) Second, E must be an integer, because Problem 11 says that Rishabh has E colors of socks.

Now, the constraint from Problem 6 allows us to do casework on the value of B. Because E is an integer, so is F, and so is A. $0 < A < 100 \implies 1 \le A \le 99 \implies 0 \le \lfloor \log_2 A \rfloor \le 6$, which means $B \in \{1, 2, 3, 4, 5, 6, 7\}$.

B	1	2	3	4	5	6	7
C	1	1	2	6	24	120	720
D	$\frac{7}{12}$	$\frac{7}{12}$	$\frac{7}{6}$	$\frac{7}{2}$	14	70	420
E	$\frac{7\sqrt{6}}{12}$	$\frac{7\sqrt{6}}{12}$	$\frac{7\sqrt{3}}{6}$	$\frac{7}{2}$	7	$7\sqrt{5}$	$7\sqrt{30}$

We can see that the only case where E is an integer is when $B = 5$. Now, we see that $F = E - 1 = 6$ and we can finally compute the value of A:

$$\begin{aligned}
&F + 2F + 3F + \cdots + F^2 \\
&= F(1 + 2 + 3 + \cdots + F) \\
&= \frac{F^2(F+1)}{2} \\
&= \frac{6^2 \cdot 7}{2} = 126,
\end{aligned}$$

which, when taken mod 100, gives $A = 26$. To check, $\lfloor \log_2 A \rfloor + 1$ is indeed 5, so we have successfully proven that this is the only set of values that works.

The answers to this section are tabulated below.

A	B	C	D	E	F
26	5	24	14	7	6

$\square$

3 I'm beginnin' to feel like a...

(1) PDA $\times$ POE = AAAAA.

(2) PIE $-$ EGG = EEA.

(3) RST + RTS + SRT + STR + TRS + TSR = R???.

(4) POTATO is divisible by 3.

Solution. In this solution, an overline will be used to refer to a number made of some number of unknown digits, like $\overline{\text{PIE}}$, equal in value to $100\text{P} + 10\text{I} + \text{E}$.

We start with statement (1), which says

$$\overline{\text{PDA}} \times \overline{\text{POE}} = \overline{\text{AAAAA}} = 11111\text{A} = \text{A} \cdot 41 \cdot 271.$$

Note that, because 41 and 271 both have a units digit of 1, $\overline{\text{PDA}}$ must be the product of A and a factor of 11111. Since both factors are three digits, $\overline{\text{POE}}$ must equal 271 and $\overline{\text{PDA}}$ must equal 41A. Specifically, because $\text{P} = 2$, we have $200 \le 41\text{A} < 300$, and A is either 5, 6, or 7. However, A cannot be 7, because O is 7, and letters must be distinct.

We now move on to statement (2), which says

$$\begin{aligned} \overline{\text{PIE}} - \overline{\text{EGG}} &= \overline{\text{EEA}} \\ \overline{\text{2I1}} - \overline{\text{1GG}} &= \overline{\text{11A}} \\ 101 + 10\text{I} - 11\text{G} &= 110 + \text{A} \\ 10\text{I} &= 11\text{G} + \text{A} + 9. \end{aligned}$$

Because the left-hand side is divisible by 10, the same must be true for the right-hand side. We now casework on what A is. If $\text{A} = 6$, then $10\text{I} = 11\text{G} + 15$, so G must be 5 and I must be 7. However, O is 7, so this is impossible. This means that $\text{A} = 5$, which then gives $\overline{\text{PDA}} = 41 \cdot 5 = 205$ and $10\text{I} = 11\text{G} + 14$, so $\text{G} = 6$ and $\text{I} = 8$.

Let's tabulate what we have so far.

Digit	0	1	2	3	4	5	6	7	8	9
Letter	D	E	P	?	?	A	G	O	I	?

We now look at statement (3), which says that the quantity

$$\begin{aligned} &\overline{\text{RST}} + \overline{\text{RTS}} + \overline{\text{SRT}} + \overline{\text{STR}} + \overline{\text{TRS}} + \overline{\text{TSR}} \\ &= (100\text{R} + 10\text{S} + \text{T}) + (100\text{R} + 10\text{T} + \text{S}) + (100\text{S} + 10\text{R} + \text{T}) \\ &+ (100\text{S} + 10\text{T} + \text{R}) + (100\text{T} + 10\text{R} + \text{S}) + (100\text{T} + 10\text{S} + \text{R}) \\ &= 222(\text{R} + \text{S} + \text{T}) \end{aligned}$$

is a four-digit number with a thousands digit of R. Note that, because we have not yet seen the letters R, S, or T, they must be the missing digits 3, 4, and 9 in some order. This order does not matter, because $222(\mathrm{R}+\mathrm{S}+\mathrm{T})$ is symmetric, and equals $222(3+4+9) = 222 \cdot 16 = 3552$, meaning $\mathrm{R} = 3$.

Finally, statement (4) tells us that POTATO is divisible by 3, which means that $\mathrm{P} + \mathrm{O} + \mathrm{T} + \mathrm{A} + \mathrm{T} + \mathrm{O} = 2 + 7 + \mathrm{T} + 5 + 7 + \mathrm{T} = 21 + 2\mathrm{T}$ is also divisible by 3. The only multiple of 3 left is 9, so $\mathrm{T} = 9$, which leaves $\mathrm{S} = 4$.

The correct correspondence between digits and letters is shown in the table below.

Digit	0	1	2	3	4	5	6	7	8	9
Letter	D	E	P	R	S	A	G	O	I	T

The value of $\overline{\mathrm{RAP}} \times \overline{\mathrm{GOD}} + 1$ is $352 \cdot 670 + 1 = \boxed{235841}$. $\square$

4 (Optional) This section is not a puzzle...

The answer to this puzzle is an English phrase, with one part extracted from each section of the homework.

Let's tabulate the answers to Section 1.

Problem	1	2	3	4	5
Answer	161	39	105	35*	90

*The answer to Problem 4 is actually 36. An email was sent on March 31st to subtract 1 from the answer to use for the puzzle. I apologize for any confusion.

These are all positive integers with nothing special about them. The most direct way to turn these into letters is to match their residues mod 26 to the alphabet. This is further clued by the section title "Same Old, Same Old", which refers to the five problem format employed by previous homeworks, but also referring to the extraction method in the puzzle from Homework #4.

Taken mod 26, the answers become 5, 13, 1, 9, and 12, in that order. These correspond to the letters E, M, A, I, and L, which produces the word $\boxed{\text{EMAIL}}$.

Let's tabulate the answers to Section 2.

Problem	1	2	3	4	5	6
Answer	26	5	24	14	17	6

These are positive integers in the interval $[1, 26]$, which correspond directly to letters of the alphabet! But, upon doing so, we get. . . ZEXNQF. That's unintelligible. But wait. It can't be that easy. Looking to the section title again, "Circular Reasoning", we see that it refers to the relay-esque structure of the section. However, it also refers to a common cipher—the Caesar Cipher, which involves "rotating" a cipher alphabet.

No specific shift is named, so we can use an online tool to quickly scan all possible shifts. It turns out that the correct shift is the ROT13 encoding, which translates ZEXNGF into $\boxed{\text{MRKATS}}$. Now we're getting somewhere. . .

Finally, the last section. There is no hint in the section title this time, but the most direct way to translate the answer into letters is to replace each digit with its corresponding letter in the problem. Then, 235841 translates to $\boxed{\text{PRAISE}}$.

The answer to the puzzle is then $\boxed{\text{EMAIL MR KATS PRAISE}}$, which is verified by an email to Mr. Kats detailing your appreciation for him.

Remark. An email only containing the word "PRAISE" was considered a partial solve.