NYCMT 2023-2024 Homework #6

NYCMT

March 30th - May 3rd, 2024

The problems that follow are due May 3rd. Please solve as many problems as you can, and write up solutions (**not just answers!**) to the ones you solve, unless otherwise noted. Write down any progress you make on problems you don't solve.

Please write solutions for different questions on separate pages, unless otherwise noted. Make sure to write your name on each page and page numbers per problem.

If you're not going to be present on May 3rd, you can scan your solutions and email them to ali40@stuy.edu, dpotievsky40@stuy.edu, and jaeminkim2@hunterschools.org. If you will be there, just hand in your responses on paper. If you have any questions, feel free to ask one of us on Discord or using one of the above emails.

Problems are NOT difficulty-ordered, so you should read and try all of them. Enjoy!

1 Same Old, Same Old

Problem 1. A fair four-sided die has faces labeled 1, 2, 3, and 4. This die is rolled six times, and the numbers rolled are multiplied together. The probability that this product is a perfect square can be expressed as $\frac{m}{n}$, where m and n are relatively prime positive integers. Find $m+n$.

Problem 2. There exists a unique positive integer n such that $36n$ has 36 factors and $48n$ has 48 factors. How many factors does n^2 have?

Problem 3. In $\triangle ABC$, D is the midpoint of $\overline{BC}$, E is the midpoint of $\overline{AC}$, and F is the midpoint of $\overline{AB}$. If $\overline{AD} \perp \overline{BE}$, $AD = 63$, and $BE = 84$, find the length of $\overline{CF}$.

Problem 4. Chenkai and Edwin are eating croffles at constant rates. Chenkai can eat an entire croffle by himself in 45 seconds, but Edwin eats even faster, and can do so in x seconds. Chenkai and Edwin each start eating their own croffle at the same time, but when Edwin is done with his, Chenkai realizes that he is eating someone else's croffle. After apologizing profusely, he buys another. Edwin is still hungry, so they both eat the third croffle. If Edwin spent a total of 56 seconds eating croffles, find x.

Problem 5. Let $\lfloor x \rfloor$ denote the greatest integer less than or equal to x. Find the ninth smallest positive integer n such that there are no real solutions to $x \cdot \lfloor x \rfloor = n$.

2 Circular Reasoning

In this section, solutions to different problems need not be on different pages, as the solution to each problem relies on those of other problems.

Problem 6. Let F be the answer to Problem 11. Find the remainder when

$$F + 2F + 3F + \cdots + F^2$$

is divided by 100.

Problem 7. Let A be the answer to Problem 6. On Day 0, there are A bananas. On each subsequent day, if there are k bananas left, Noam eats $\lceil \frac{k}{2} \rceil$ of them. On what day does Noam eat the last banana?

Problem 8. Let B be the answer to Problem 7. There are B friends who want to participate in a game of assassin, in which each player is assigned a target to eliminate. When a player is eliminated, their target becomes the new target of their eliminator. (Players cannot eliminate themselves.) Once there are no more possible eliminations, there is one player left. How many initial assignments of targets were possible?

Problem 9. Let C be the answer to Problem 8. A right triangle with perimeter C has side lengths that form an arithmetic progression. Find the sum of the lengths of its legs.

Problem 10. Let D be the answer to Problem 9. A rectangle with area D has diagonals that intersect at an angle θ, with $\sin\theta = \frac{1}{7}$. This rectangle is inscribed in a circle of radius r. Find r.

Problem 11. Let E be the answer to Problem 10. Rishabh has E colors of socks, and E socks of each color, for a total of E^2 socks in his drawer. He removes socks one at a time until he is absolutely certain that he has at least one sock of each color. How many socks are left in the drawer?

3 I'm beginnin' to feel like a...

This section is **Answer Only**; you do not need to provide a solution.

Each letter represents a distinct digit between 0 and 9 inclusive.

When multiple letters or numbers are written in a row, they represent the individual digits of a larger number, **not** a product of variables. For example, PIE denotes a three-digit number equal to $100 \cdot \mathrm{P} + 10 \cdot \mathrm{I} + \mathrm{E}$.

Leading variables are never 0. For example, because PIE has the leading variable P, we can conclude that $\mathrm{P} \neq 0$.

Question marks (?) indicate letters that are unknown. For example, R??? denotes a four-digit number with a thousands digit of R.

1. $\mathrm{PDA} \times \mathrm{POE} = \mathrm{AAAAA}$.
2. $\mathrm{PIE} - \mathrm{EGG} = \mathrm{EEA}$.
3. $\mathrm{RST} + \mathrm{RTS} + \mathrm{SRT} + \mathrm{STR} + \mathrm{TRS} + \mathrm{TSR} = \mathrm{R???}$.
4. POTATO is divisible by 3.

First, determine the unique value of each letter. Then, find the value of $\mathrm{RAP} \times \mathrm{GOD} + 1$.

4 (Optional) This section is not a puzzle...

... but the other three sections are!

If you solve the puzzle correctly, you will know how to confirm your answer.

The first person to solve the puzzle gets a *really good* donut!

If no one has solved the puzzle by April 26th, a global hint will be emailed.