NYCMT 2022-2023 HW#1

NYCMT

September 16 - September 23, 2022

1 Solutions

Problem 1. The equation $742586 + 829430 = 1212016$ is incorrect. However, we can change one digit d in the equation to another e to create a true equation. Find, with proof, the ordered pair (d, e).

Answer: $(2, 6)$
Solution: Firstly we should note that $742586 + 829430$ is actually equal to 1572016. Now lets highlight the digits involved in any incorrect sums.

$$\begin{array}{cccccccc} & & 7 & 4 & 2 & 5 & 8 & 6 \\ & + & 8 & 2 & 9 & 4 & 3 & 0 \\ \hline & 1 & 2 & 1 & 2 & 0 & 1 & 6 \end{array}$$

Now we want to look for a digit we can change that will fix both these sums simultaneously. 2 is an excellent candidate because it occurs in both columns. Now lets figure out what digit x we should replace 2 with.

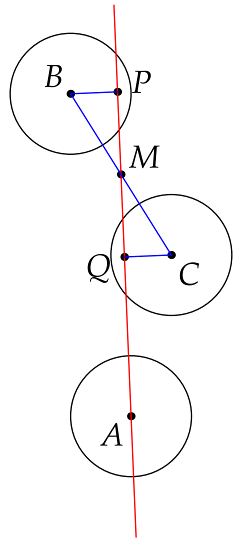
$$\begin{array}{cccccccc} & & 7 & 4 & x & 5 & 8 & 6 \\ & + & 8 & x & 9 & 4 & 3 & 0 \\ \hline & 1 & x & 1 & x & 0 & 1 & 6 \end{array}$$

Once we account for all possible carryovers, we get that $x + 9 + 1$ ends in the digit x, $4 + x + 1$ ends in the digit 1, and $7 + 8 + ?? = x$ where the ?? represents the possibility of a carryover in the sum $4 + x + 1$.
If $4 + x + 1$ ends in the digit 1, then we know $4 + x + 1 = 11$ and so $x = 6$. Indeed this does make the equation true, so the solution is $(2, 6)$

Now just to be explicit, we will explain why no choice other than 2 can work. Firstly, we cannot change the digit 1 because we know the first digit of the result must be a 1. We cannot change the digit 3 because it is involved in a currently-correct sum that will become incorrect. For similar logic we cannot change 0, 4, 5, 6, 8, and 9. This leaves only 7 or 2 to be changed. Changing 7 will not correct the column involving 4, 2 and 1 and so the only choice for what digit to change is 2.

Problem 2. Three circles of radius 3 have centers $(2, 22)$, $(5, 6)$, and $(7, 14)$, respectively. A line passing through $(5, 6)$ is drawn such that the sum of the areas of the parts of the circles on each side of the line are equal. What is the y-intercept of this line?

Answer: 126
Solution: Let $A = (5, 6)$, $B = (2, 22)$ and $C = (7, 14)$.
Let P and Q be the feet from B and C to our line.
Let M be where BC meets PQ.



Note that since our line passes through the center of the circle centered at A it automatically splits it into 2 equal parts. Thus the line must split the circles centered at B and C symmetrically.
Thus the line must be equidistant from B and C.

This tells us that $BP = QC$. Furthermore, $\angle BPM = \angle QCM = 90$, and using vertical angles we know that $\angle BMP = \angle CMQ$. Thus $\triangle BMP \cong \triangle CMQ$.

Now we know our line passes through the midpoint of BC (since $BM = MC$ and M is on the line), and through A. This determines our line, so now we just need to compute the y intercept.

$M = (\frac{2+7}{2}, \frac{22+14}{2}) = (\frac{9}{2}, 18)$. Thus our line has equation $y - 6 = \frac{18-6}{\frac{9}{2}-5}(x - 5)$, or $y = -24x + 126$, so our final answer is 126.

Problem 3. Let $P(x) = x^6 + 8x^5 + 12x^4 + 3x^3 + 7x^2 + 14x + 2$ be a polynomial with roots $r_1, r_2, \cdots r_6$. Compute

$$\sum_{1 \le i < j \le 6} \frac{(r_i + r_j)^2}{r_i r_j}.$$

Answer: 80

Solution: Note that $\frac{(r_i+r_j)^2}{r_i r_j} = \frac{r_i^2 + 2r_i r_j + r_j^2}{r_i r_j} = \frac{r_i}{r_j} + \frac{r_j}{r_i} + 2$.
Thus our sum reduces to

$$\sum_{1 \le i < j \le 6} \frac{r_i}{r_j} + \frac{r_j}{r_i} + 2$$

There are $\binom{6}{2}$ choices for i, j, and so we can pull out the 2 from the summation and include it as $2\binom{6}{2} = 30$.

Looking at the sum of $\frac{r_i}{r_j} + \frac{r_j}{r_i}$, we see every fraction of the form $\frac{r_a}{r_b}$ appears exactly once, except for when $a = b$. Adding and subtracting in the fractions of the form $\frac{r_a}{r_a}$ (of which there are 6) we see that we have a sum where each value of r_a is paired with every possible value of $\frac{1}{r_b}$.

Thus we can rewrite the expression as

$$(\sum_{1 \le a \le 6} r_a)(\sum_{1 \le b \le 6} \frac{1}{r_b}) - 6$$

The minus 6 comes from when we added and subtracted 6.

Using vietas, we see $\sum\limits_{1 \le a \le 6} r_a = -8$, and $\sum\limits_{1 \le b \le 6} \frac{1}{r_b} = \frac{\text{Sum of roots taken 5 at a time}}{\text{Product of all roots}} = \frac{-14}{2}$.
Thus our final answer will be $(-8)(\frac{-14}{2}) - 6 + 30 = 80$.

Remark: There were many cool ways to rewrite the expression in terms of vieta's. TALK TO THE PEOPLE AROUND YOU to hear how they did it!

Problem 4. Compute the number of triples of positive integers (a, b, c) such that $18!|a|b|c|21!$.

Answer: 2560
Solution: Since $18!|a, b, c$ we can actually write a, b, c as $18!\hat{a}, 18!\hat{b}, 18!\hat{c}$.
Notice that in order to force $18!\hat{a}, 18!\hat{b}, 18!\hat{c}|21!$, $\hat{a}, \hat{b}, \hat{c}$ must be among the divisors of $\frac{21!}{18!} = 19(20)(21) = 2^2(3)(5)(7)(19)$

Now we need to ensure that $18!\hat{a}|18!\hat{b}|18!\hat{c}$. Since each number has a factor of 18!, this is equivalent to $\hat{a}|\hat{b}|\hat{c}$.
Noting that each of $\hat{a}, \hat{b}, \hat{c}$ looks like $2^m(3^n)(5^p)(7^q)(19^r)$ where $0 \le m \le 2$, $0 \le n \le 1$, $0 \le p \le 1$, $0 \le q \le 1$, $0 \le r \le 1$ we can force the dividing condition by picking three values of m, n, p, q, r that look like $m_a, m_b, m_c, n_a, n_b, n_c$ and so on so that $m_a \le m_b \le m_c$ and $n_a \le n_b \le n_c$.
Thus $\hat{a}$ will equal $2^{m_a}(3^{n_a})(5^{p_a})(7^{q_a})(19^{r_a}$ and the others will be defined similarly.
The condition that $m_a \le m_b \le m_c$ ensures that all powers of 2 in $\hat{a}$ will divide the powers of 2 in $\hat{b}$ and those will divide the powers of 3 in $\hat{c}$. Repeating this we know that all the prime powers in $\hat{a}$ divide the prime powers in $\hat{b}$ which in turn divide the prime powers in $\hat{c}$, forcing $\hat{a}$ to divide $\hat{b}$ to divide $\hat{c}$.

Now to computation! If $0 \le m_a \le m_b \le m_c \le 2$, we can remove the "equals-to" by seeing that $0 \le m_a < m_b + 1 < m_b + 2 \le 4$. Thus we want to pick 3 distinct numbers from 0 to 4, order them from least to greatest, and subtract 1 from the second and 2 from the third to give us a triple of m_a, m_b, m_c. There are $\binom{5}{3}$ ways to pick 3 distinct numbers, leading to 10 unique triples of m_a, m_b, m_c.

Repeating this for all the remaining variables we get a final answer of $\binom{5}{3}\binom{4}{3}\binom{4}{3}\binom{4}{3}\binom{4}{3} = 2560$.

Remark: We can also compute the number of triples m_a, m_b, m_c as follows: Consider 2 stars, and 3 bars. Denote m_a as the number of stars to the left of the left most bar, m_b as the number of stars to the left of the middle bar, and m_c as the number of stars to the left of the right most bar. Doing this will ensure $m_a \le m_b \le m_c$ and that all three are between 0 and 2, and thus will give us the desired value. There are $\binom{5}{3}$ ways to arrange 2 stars and 3 bars and thus $\binom{5}{3}$ triples.

Remark: Another solution method is to let $p = \frac{a}{18!}$, $q = \frac{b}{a}$, $r = \frac{c}{b}$, $s = \frac{21!}{c}$. Note that p, q, r, s must all be integers, and $pqrs = \frac{21!}{18!}$. Thus we need to split up the $\frac{21!}{18!} = 2^2(3)(5)(7)(19)$ into four factors, which can be done by distributing the two 2s, one 3, one 5, one 7 and one 19 into four groups. Stars and bars tells us we need three bars to create four groups, and so we get a final answer of $\binom{2+3}{3}\binom{1+3}{3}\binom{1+3}{3}\binom{1+3}{3}\binom{1+3}{3}$

Problem 5. Define the Fibonacci sequence as follows, $F_1 = 1$, $F_2 = 1$ and $F_n = F_{n-1} + F_{n-2}$ for all $n > 2$. Find all solutions j for which

(a) $F_j = j$

(b) $F_j = j^2$

Answer: (a) $j = 1, 5$ (b) $j = 1, 12$
Solution: Making a list of the first 13 Fibonacci numbers we get

n	1	2	3	4	5	6	7	8	9	10	11	12	13
F_n	1	1	2	3	5	8	13	21	34	55	89	144	233

We can see that $j = 1$ and $j = 5$ satisfy the condition that $F_j = j$. It looks like that for all values of $n > 5$ $F_n > n$. This is indeed the case, and we can show this with induction.

Claim: For all $n > 5$ $F_n > n$
Base Case: When $n = 6$, $F_6 = 8 > 6$.
Inductive Hypothesis: Assume that for all $5 < l \leq k$ for some $k > 5$ $F_l > l$.
Inductive Step: Note that $F_{k-1} > 1$. Thus since $F_k > k$ (by the Inductive Hypothesis),

$$F_{k+1} = F_k + F_{k-1} > k + 1$$

Thus $j = 1$ and $j = 5$ are the only possible solutions to part (a).

We can also see that $j = 1$ and $j = 12$ satisfy the condition $F_j = j^2$. We will use a similar inductive argument to show that for all values of $n > 12$ $F_n > n^2$.

Claim: For all $n > 12$ $F_n > n^2$
Base Case: When $n = 13$, $F_13 = 233 > 169$.
Inductive Hypothesis: Assume that for all $12 < l \leq k$ for some $k > 12$ $F_l > l^2$.
Inductive Step: Utilizing an argument similar to the one used in (a) we can see that $F_k > 2k + 1$ for all $k > 12$. Thus since $F_k > k^2$ (by the Inductive Hypothesis),

$$F_{k+1} = F_k + F_{k-1} > k^2 + 2k + 1 = (k+1)^2$$

Thus $j = 1$ and $j = 12$ are the only possible solutions for part (b).

Remark: We can rationalize the approaches used in both (a) and (b) by using the formula for the n^{th} Fibonacci number:

$$F_n = \frac{(\frac{1+\sqrt{5}}{2})^{n-1} - (\frac{1-\sqrt{5}}{2})^{n-1}}{\sqrt{5}}$$

Note that this formula for F_n is *geometric* in nature. It will therefor grow much faster than a polynomial like x^2, and so it is more clear that there must be a certain point where $F_n > n^2$.

Problem 6. Prove or disprove: every positive integer at least 6 can be written as the sum of 3 primes.